Chow Theory and Milnor K-theory

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Abstract. In this paper, we study Chow theory with coefficients in Rost's cycle modules. In particular, we demonstrate that Milnor K-theory is an instance of cycle module, and we show how this perspective naturally generalizes the classical presentation of Chow groups into a full cycle complex.

Keywords. Chow groups, Milnor K-theory, cycle modules.

Contents

1. Introduction

Algebraic geometry investigates geometric phenomena within a broad and abstract framework using the tools of algebra. A variety of sophisticated techniques have been developed to address classical

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problems or to reformulate them in more tractable algebraic terms. Among these, Chow groups play a central role, serving as fundamental invariants with wide-ranging applications across the field. Originally introduced by Francesco Severi in the 1930s and substantially refined by Wei-Liang Chow in the 1950s, Chow groups were further developed through the influential work of William Fulton and Robert MacPherson in the 1970s. These groups can be viewed as an early attempt to incorporate homotopical methods into algebraic geometry, preceding the advent of more modern tools such as algebraic K-theory.

Chow groups are intended to study the foundational objects of algebraic geometry -— algebraic varieties and, more generally, schemes – by encoding information about their algebraic subvarieties. These groups are generated by algebraic cycles, analogous to how cellular homology is generated by cells or subcomplexes in a CW-complex. Namely, for a nice scheme or variety X of pure dimension $m \geq n$, its n-th Chow group is

$$CH_n(X) = Z_n(X)/\sim_n, \qquad \text{(Definition 2.32)}$$

where $Z_n(X)$ is the abelian group of n-cycles (Definition 2.29), and $\sim_i$ is generated by the rational equivalence (Definition 2.30). For smooth varieties, the Chow groups possess additional structure: they admit an intersection product that endows the collection of Chow groups with the structure of a graded ring, known as the Chow ring. This construction bears a strong analogy to the cohomology ring in classical topology, providing a bridge between intersection theory in algebraic geometry and homological methods in topology. For example, the line bundles over a scheme X of pure dimension n can be characterized by its $(n-1)$-th Chow groups, see Theorem 2.28.

There are various approaches to defining the Chow groups (and Chow rings) of a smooth scheme X of pure dimension n, beyond the classical description in terms of algebraic cycles modulo rational equivalence. One notable alternative is the presentation of Chow groups via a complex of abelian groups. Specifically, for each $i \leq n$, there is a short exact sequence

$$\coprod_{y \in X_{(i+1)}} \kappa(y) \to \coprod_{y \in X_{(i)}} \mathbb{Z} \to CH_i(X) \to 0, \tag{3.4}$$

where $X_{(i)}$ is the dimension i subvariety of X, and $\kappa(y)$ is the residue field at y. This presentation naturally raises the question:

Question 1.1. Can the factors in the coproducts (3.4) be modified?

A compelling consequence of such a modification is the potential to define Chow groups with coefficients. In classical algebraic topology, it is common to replace $\mathbb{Z}$-coefficient in singular cohomology with other rings in order to detect certain cohomological operations, such as Steenrod operations. A similar enrichment in the context of Chow theory could lead to new insights, allowing one to extract additional geometric or arithmetic information from algebraic varieties by choosing appropriate coefficient rings.

The answer to the above question is yes. Indeed, for a certain class of coefficient systems -— namely, cycle modules M (Definition 3.35) —- one can define Chow groups with coefficients M. The notion of cycle modules was first introduced by Markus Rost in his work 1996 [Ros96], motivated in part by the desire to generalize classical Chow groups in Question 1.1 and to develop an intersection theory compatible with these enriched structures. The formal construction of cycle modules is rather complicated, and will be presented in in §3.3.

A key example of a cycle module is Milnor K-theory, which will be discussed in detail in §3.1. Introduced by Milnor in [Mil70], Milnor K-theory was conceived as a first attempt to capture aspects of higher algebraic K-theory, particularly for fields. While its construction may now appear somewhat ad hoc, especially in light of the modern definitions of higher K-groups, it remains of considerable interest for number theorists and algebraists. Milnor K-theory offers a computational approach to étale cohomology of fields and plays a central role in the proof of the celebrated Bloch–Kato conjecture, now a theorem due to Voevodsky.

More precisely, there exists a norm residue map from Milnor K-theory of a field k to the étale cohomology of $\operatorname{Spec} k$ in μ_ℓ-coefficient. Upon passing to motivic cohomology, Rost and Voevodsky independently showed that both sides of this map are isomorphic to the same motivic cohomology group, thereby establishing the norm residue isomorphism. When $\ell = 2$, this result recovers the celebrated classical Milnor conjecture. As a result, the Lichtenbaum–Quillen conjecture follows from the Bloch–Kato conjecture via the motivic spectral sequence associated with algebraic K-theory.

We will not explore these developments further in this article. For additional details and broader context, the reader is referred to the author's survey notes [Yan25].

As a consequence of the general framework of cycle modules, one obtains the following presentation of Chow groups via Milnor K-theory:

$$\coprod_{y \in X_{(i+1)}} \mathrm{K}_1^M(\kappa(y)) \xrightarrow{\partial_y} \coprod_{y \in X_{(i)}} \mathrm{K}_0^M(\kappa(y)) \to CH_i(X) \to 0. \tag{3.5}$$

This exact sequence not only recovers the classical Chow groups, but can also be prolonged to both sides into a chain complex, known as the cycle complex, which encodes higher-dimensional intersection data. Such a construction is valid for any cycle module and provides a unifying framework for understanding cohomological invariants in algebraic geometry. Although a detailed treatment of the intersection theory arising from these cycle complexes lies beyond the scope of this paper, we will outline a proof that Milnor K-theory satisfies the axioms of a cycle module. This ensures that all subsequent constructions are well-defined within Rost's framework:

Theorem 1.2 (c.f. Theorem 3.37)**.** *Milnor K-theory functor* $\mathrm{K}_*^M(-)$ *is a cycle module.*

How to read this paper. This paper is organized into four chapters, including the introduction.

Chapter §2 provides a general introduction to Chow theory. We begin by reviewing the necessary background in algebraic geometry in §2.1 and §2.2. The construction and basic properties of Chow groups and Chow rings are developed in §2.3 and §2.4. The chapter concludes with a brief comparison between Chow theory and classical Borel–Moore homology (also known as locally finite homology) in §2.5.

Chapter §3 forms the core of the paper. We introduce Milnor K-theory and its foundational properties in §3.1, including a brief review of relevant background in algebraic number theory. In §3.2 and §3.3, we present Rost's theory of cycle modules, following his work in [Ros96], along with the structural properties that make this theory suitable for generalizing classical intersection theory.

Chapter §4 focuses on cycle complexes and their properties. In §4.3, we provide a sketch of the proof of Theorem 3.37, ensuring that the constructions appearing in earlier sections are rigorously grounded.

This paper is intended to be self-contained. Readers familiar with scheme theory and algebraic geometry may wish to begin directly with §2.3 for a quick review, and then proceed to Chapter §3.

Readers with a background in number theory might find it more convenient to skip Chapter §2 and start at §3.2. Those primarily interested in the proof of Theorem 3.37 may proceed directly to §4.3, referring back to §3.2 and §3.3 as needed. In any case, it is suggested that first-time readers omit §2.5, as it is not essential to the main development of the paper.

2. Chow theory

In this chapter will deal with the basics in Chow theory. For the rest of the chapter, we will fix an arbitrary field k.

2.1. **Prerequisites of schemes.** Before we move on, we will quickly collect some basic terminology in the algebraic geometry. Readers familiar with the subject may wish to proceed directly to §2.3.

Definition 2.1. A **scheme** X is a locally ringed space $(\underline{X}, \mathcal{O}_X)$, where $\underline{X}$ is the underlying topological space, and $\mathcal{O}_X$ is the structure sheaf, such that this pair is locally affine, i.e. $\underline{X}$ is covered by open sets U_i such that there exists rings R_i with $\mathcal{O}_X\mid_{U_i} \cong \operatorname{Spec}(R_i)$.

Definition 2.2. A scheme X is **reduced** if every local ring $\mathcal{O}_{X,x}$ is reduced, for $x \in \underline{X}$. That is, every nilpotent element in this local ring is zero. X is **irreducible** if $\underline{X}$ cannot be written as a union of two proper closed subsets.

For the rest of the paper, we will abuse the notation of a scheme X and its underlying topological space $\underline{X}$ when there is no confusion raised. A morphism of schemes $f : X \to Y$ is just a pair $(\underline{f}, f^{\#})$, where $\underline{f}$ is a continuous map of underlying spaces $\underline{f} : \underline{X} \to \underline{Y}$, and $f^{\#} : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is the map of structure sheaves.

Definition 2.3. A **k-scheme** is a scheme X endowed with a morphism $X \to \operatorname{Spec} k$. In general, one can replace $\operatorname{Spec} k$ by some other scheme S.

If it is not specified what scheme X is over, then X is automatically considered to be a scheme on $\operatorname{Spec} \mathbb{Z}$.

Definition 2.4. Let $f : X \to Y$ be a morphism of schemes. f is called **open immersion** (resp. **closed immersion**) if $f(X)$ is open (resp. closed) subset of Y, $\underline{f}$ is a homeomorphism onto the image, and $f^{\#}$ is an isomorphism (resp. surjective) restricted to the image.

Definition 2.5. Let X be a scheme. A **closed subscheme** Y of X is a locally ringed space $(\underline{Y}, \mathcal{O}_Y)$, where $\imath : \underline{Y} \hookrightarrow \underline{X}$ is a closed subspace, and $\imath_*\mathcal{O}_Y \cong \mathcal{O}_X/\mathcal{I}$, where $\mathcal{I}$ is the quasi-coherent sheaf of ideals.

Definition 2.6. Let $f : X \to Y$ be a morphism of schemes. f is called **projective** if it factors into a closed immersion $\imath : X \to \mathbb{P}^n_Y$ for some n, followed by the projection $\jmath : \mathbb{P}^n_Y \to Y$. Here $\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} Y$ is the projective n-space over Y.

Definition 2.7. A scheme X is called **quasi-compact** if the underlying space is quasi-compact, i.e. any open covering of X has a finite subcovering.

Definition 2.8. A map of schemes $f : X \to Y$ is called **quasi-separated** (resp. **separated**) if the diagonal map $X \to X \times_Y X$ is quasi-compact (resp. closed immersion), i.e. the inverse image of any quasi-compact open set is quasi-compact. A scheme X is **quasi-separated** (resp. **separated**) if $X \to \operatorname{Spec} \mathbb{Z}$ is quasi-separated (resp. a closed immersion).

We write Sch_S to be the category of S-schemes, where S is itself a scheme.

Definition 2.9. Let $X \in \mathsf{Sch}_S$. It is of **finite type** if $f : X \to S$ satisfies the following conditions:

(a) f is quasi-compact, and
(b) for any affine open subset $V \subset S$, and any affine open subset $U \subset f^{-1}(V)$, the canonical homomorphism $\mathcal{O}_S(V) \to \mathcal{O}_X(U)$ makes $\mathcal{O}_X(U)$ into a finitely generated $\mathcal{O}_S(V)$-algebra.

Definition 2.10. A scheme X is **noetherian** if it is quasi-compact and locally noetherian, i.e. it can be covered by a family U_i of open subsets of the form $U_i = \operatorname{Spec} R_i$, where R_i is noetherian.

For the rest of the paper, we will always assume a k-scheme is separated, noetherian, and of finite type. Some examples that will be frequently used including $\mathbb{A}^n_k$, $\mathbb{G}_m$, $\mathbb{P}^n_k$, and (separated) quasi-projective k-schemes (means $f : X \to \operatorname{Spec} k$ factors into an open immersion $g : X \to X'$ followed by a projective morphism $h : X' \to \operatorname{Spec} k$).

Definition 2.11. A scheme X is **normal** if for all $x \in X$, $\mathcal{O}_{X,x}$ is a domain which is integrally closed in its field of fractions.

Definition 2.12. A k-scheme X is **proper** if X is of finite type, and the structure map $f : X \to \operatorname{Spec} k$ is separated and universally closed, i.e. for every map of schemes $g : Y \to \operatorname{Spec} k$, the pullback $Y \times_{\operatorname{Spec} k} X \to Y$ is closed.

Definition 2.13. A **variety** over k is an integral (reduced and irreducible) k-scheme. Equivalently, for every affine open subset $U \subset X$, $\mathcal{O}_X(U)$ is an integral domain.

Example 2.14. The most naive examples of varieties over k are the affine varieties. In short, affine varieties are the closed subvarieties (as subschemes) of $\mathbb{A}^n_k$. They are in one-to-one correspondence with the ideals in $k[x_1, x_2, \cdots, x_n]$ by Hilbert's Nullstellensatz. Thus, we can write an affine variety as $X = \{f_1 = 0, \cdots, f_r = 0\} \subset \mathbb{A}^n_k$ for $\{f_1, \cdots, f_r\}$ generating an ideal $I \subset k[x_1, x_2, \cdots, x_n]$ (each f_i has no multiplicities), with $\mathcal{O}_X(X) = k[x_1, x_2, \cdots, x_n]/I$.

Remark 2.15. The difference between schemes and varieties are as follows:

(a) Schemes might have multiple irreducible components, in analogy to the topological spaces with multiple connected components.
(b) Schemes might be non-reduced, especially when considering the multiplicities. For example, $f(x) = (x-2)^2 \in \mathbb{C}[x]$, and $X = \{x \in \mathbb{C} : f(x) = 0\}$ be the closed subscheme of the affine line $\mathbb{A}^1_{\mathbb{C}}$, $\mathcal{O}_X(X) = \mathbb{C}[x]/(x-2)^2$ is then not reduced. Thus $(X, \mathcal{O}_X)$ is not a variety, while it is indeed a scheme.

For the purpose of our paper, we end this section by introducing the concept of smooth schemes.

Definition 2.16. Let X be a k-scheme of finite type. X is **smooth over** k if $f : X \to \operatorname{Spec} k$ is smooth. That is, f satisfies

(a) f is flat,
(b) f is locally of finite presentation, and
(c) for every $y \in \operatorname{Spec} k$, the localizations of the ring $\mathcal{O}_{X,f^{-1}(y)} \otimes_{\mathcal{O}_{Y,y}} \overline{k(y)}$ are regular. where $\overline{k(y)}$ is the algebraic closure of the residue field of $\mathcal{O}_{Y,y}$.

In the case X is a variety over k, X is smooth if and only if it is a non-singular variety, i.e. $\mathcal{O}_{X,x}$ is regular for every $x \in X$.

Example 2.17. If X is an affine variety over k, then $X = \{f_1, \cdots, f_r = 0\} \subset \mathbb{A}_k^n$ for some $f_1, \cdots, f_r \in k[x_1, \cdots, x_n]$. By definition, X is smooth if it is a non-singular variety. This means for every point $x \in X$, one can consider the Jacobian at the x, denoted $J_x = (\frac{\partial f_j}{\partial x_i})_{i,j}$. If X is of dimension d, then X is non-singular if rank $J_x = n - d$.

Remark 2.18. There are different definitions of smoothness. In [Har13, III. §10], the smoothness is defined in a way that one replaced (3) in Definition 2.16 by asking $\dim_{k(x)}(\Omega_{X/Y} \otimes k(x)) = n$ for some positive integer n, where $x \in X$ and $\Omega_{X/Y}$ is the Kähler differential. The proof of equivalence can be seen in [Bru07, p. 7].

2.2. **Divisors.** From now on, we narrow our attention to smooth schemes. Divisors on a scheme X, in short, are codimension 1 closed subvarieties. They are of crucial roles when one tries to understand the line bundles over the scheme X, where the latter is central to algebraic geometry. For example, let L be a line bundle over X, then the zero locus of a regular section $s : X \to L$ is actually a special divisor (Theorem 2.28). For our purpose of the paper, divisors are essentially when we try to define the equivalence classes in the Chow groups.

Definition 2.19. Let X be a scheme. A **prime divisor** on X is a closed subvariety Y of codimension 1. A **(Weil) divisor** is a formal finite sum $D = \sum_i n_i Y_i$, where each Y_i is a prime divisor, and each n_i is a integer. If all $n_i \geq 0$, then D is called **effective**.

Remark 2.20 (Cartier divisors)**.** There is another notion of divisor, known as the **Cartier divisor**. For a scheme X, a Cartier divisor on X is a global section of the quotient sheaf $\mathcal{R}_X^*/\mathcal{O}_X^*$, where $\mathcal{R}_X$ is the sheaf of rational functions on X. In other words, a Cartier divisor is specified by an open cover U_i and a collection of nonzero rational functions f_i, such that f_i/f_j is a nowhere zero regular section of $\mathcal{O}_X^*$ over the overlap $U_i \cap U_j$. A Cartier divisor D is called **effective**, if there is a cover U_i , such that D is represented with $f_i \in \Gamma(U_i, \mathcal{O}_{U_i})$, for all i. As we shall see later, the set of Weil divisors forms an abelian group, and one can define a notion of rational equivalence among them. Similarly, the Cartier divisors form an abelian group, and a corresponding notion of rational equivalence can be defined in this context as well. Informally, a Cartier divisor is simply a Weil divisor defined locally by one equation. A detailed treatment of Cartier divisors lies beyond the scope of this paper, so we shall restrict our attention only to Weil divisors.

Lemma 2.21 ([Har13, II. Proposition 6.11])**.** *Let X be a variety, all of whose local rings are UFDs (in which case we say X is **locally factorial**). Then the group of Weil divisors on X is isomorphic to the group of Cartier divisors on X.*

By Lemma 2.21, in the case of nice schemes, we can use the notions of Weil divisors and Cartier divisors interchangeably. Informally, a Cartier divisor is simply a Weil divisor defined locally by one equation. A detailed treatment of Cartier divisors lies beyond the scope of this paper, so we shall restrict our attention only to Weil divisors. From now on, "divisors" will refer to Weil divisors, unless otherwise specified.

Let Y be a prime divisor of X, and $y \in Y$ be a generic point. Then $\mathcal{O}_{X,y}$ is the direct limit of the rings of regular functions on a open set $U \subset X$ containing at least one point of Y. This a local ring with a unique maximal ideal $\mathfrak{m}_y$. If X is a variety, then $\mathcal{O}_{X,y}$ can also be regarded as the ring of rational functions on X that is regular at y. Let $\kappa(y) = \mathcal{O}_{X,y}/\mathfrak{m}_y$ be the residue field. Note that $\mathcal{O}_{X,y}$ is a noetherian local ring by assumption, and regular by smoothness. It is of Krull dimension

1 because we have $\operatorname{codim}(Y, X) = \dim \mathcal{O}_{X,y} = \inf_{y' \in Y} \dim \mathcal{O}_{X,y'}$, see [GW10, §5.8]. Therefore, $\mathcal{O}_{X,y}$ is then a DVR with a discrete valuation $v_Y : \mathcal{O}_{X,y} \to \mathbb{Z}$ by [Har13, I. Theorem 6.2A]. This valuation v_Y is given by, for every non-zero $f \in \mathcal{O}_{X,y}$, $v_Y(f) := \ell(\mathcal{O}_{X,y}/(f))$, where ℓ denotes the length.

For every nonzero rational function $f = g/h$ on X for $g, h \in \mathcal{O}_{X,y}$, one can define $v_Y(f) = v_Y(g) - v_Y(h)$. If $v_Y(f) > 0$, we say f has a *zero* along Y of order $v_Y(f)$; if $v_Y(f) < 0$, we say f has a *pole* along Y of order $-v_Y(f)$.

Definition 2.22. Let f be a nonzero rational function on X. The **divisor** of f, denoted by (f), is give by

$$(f) := \sum_{\substack{Y \subset X \\ Y \text{ prime divisor}}} v_Y(f) \cdot Y.$$

Any divisor that is equal to the divisor of a function is called a **principle divisor**.

Proposition 2.23. *The definition above is well-defined, i.e. (f) is indeed a divisor.*

Proof. It suffices to show (f) is a finite summation. Equivalently, $v_Y(f) = 0$ for almost all prime divisor Y. Let $U = \operatorname{Spec} R \subset X$ be an affine open subset on which f is regular. Then $X \backslash U$ is closed and proper, and contain at most finitely many prime divisors because all others will meet U by the assumption of X being quasi-compact. On the other hand, assume $v_Y(f) \geq 0$ for any prime divisor Y on U. The case $v_Y(f) > 0$ happens if and only if Y is contained in a closed subset of U defined by the ideal $Rf \subset R$. Since $f \neq 0$, such closed subset is proper, hence contains only finitely many closed irreducible subsets of codimension 1 of U since R is noetherian. Thus, $v_Y(f) > 0$ for only finitely many Y. The case $v_Y(f) \leq 0$ is similar. $\square$

Definition 2.24. Let D, D' be two divisors on X. They are **linearly equivalent** if $D - D' = (f)$ for some rational function f. In this case, we write $D \sim D'$.

Let $\operatorname{Div}(X)$ be the set of all divisors on X. We equip this set with a group structure by setting the group operation to be the formal summation, and the unit to be 0. Write $\operatorname{Cl}(X) = \operatorname{Div}(X)/\sim$, where $\sim$ is given by the linearly equivalence.

Example 2.25. If $X = \mathbb{A}^n_k$, then $\operatorname{Cl}(X) = 0$. This is because by Gauss' lemma, $k[x_1, \cdots, x_n]$ is a UFD, so every prime ideal of height 1 is principle, by [Har13, I. Theorem 6.2A]. Note that every prime divisor Y of X corresponds to some $k[x_1, \cdots, x_n]/I$ where I is a height 1 ideal, hence principle, and is generated by a irreducible polynomial f (see Example 2.14). It is clear that $(f) = 1 \cdot Y$. Thus any linear combination of prime divisors is the linear combination of the defining polynomials of those prime divisors, yielding $\operatorname{Cl}(X) = 0$.

Moreover, one can show for R a noetherian UFD, then $\operatorname{Cl}(\operatorname{Spec} R) = 0$, in the same manner.

Example 2.26. Let $U = X \backslash Z$ for a closed proper subset Z of X. Consider any prime divisor Y on X, $Y \cap U$ is then either empty or a prime divisor on U. So any regular function f on X, the divisor (f) induces one on U by $(f)_U = \sum v_{Y \cap U}(f\mid_U) \cdot (Y \cap U)$, thus induces a surjective homomorphism $\phi : \operatorname{Cl}(X) \to \operatorname{Cl}(U)$. Since $\ker \phi = \{D = \sum n_i Y_i \in \operatorname{Cl}(X) : Y_i \subset Z, \forall i\}$, if Z is a codimension 1 subvariety, then $\ker \phi = \langle 1 \cdot Z \rangle \cong \mathbb{Z}$. Thus we have a short exact sequence

$$\mathbb{Z} \to \operatorname{Cl}(X) \to \operatorname{Cl}(U) \to 0.$$

Example 2.27. Let $X = \mathbb{P}^n_k$. Since $\mathbb{A}^n_k = X \backslash Z$, where $Z = \mathbb{P}^{n-1}_k$ is a closed proper subvariety of codimension 1. By Example 2.25 and 2.26, $\operatorname{Cl}(X) \cong \mathbb{Z}$ is generated by $\mathbb{P}^{n-1}_k$.

We end this section by introducing the relationship of line bundles over X and divisors on X, as promised at the beginning. Let $\mathrm{Pic}(X)$ be the abelian group of isomorphism classes of line bundles over X. This is called the *Picard group*, with group action given by the tensor product of line bundles. Namely, if L_1, L_2 are two line bundles over X not in the same isomorphism class, then $[L_1] + [L_2] = [L_1 \otimes L_2]$.

Theorem 2.28. *For any smooth scheme X, there is a canonical homomorphism known as the* ***first Chern class****:*

$$c_1 : \mathrm{Pic}(X) \to \mathrm{Cl}(X).$$

If X is furthermore locally factorial (see Lemma 2.21), then c_1 is an isomorphism.

Proof. Let L be a line bundle over X, and $\alpha \in \Gamma(X, L \otimes_{\mathcal{O}_X} \mathcal{R}_X)$ be a global section, where $\mathcal{R}_X$ is the sheaf of rational functions on X. We denote by α the *rational global section*. Therefore, locally on some affine open $U \subset X$, we can write $\alpha = f_U/g_U$, and so $(\alpha \mid_U) = (f) - (g)$ as divisors. This actually specifies a Cartier divisor, see Remark 2.20. We define

$$c_1(L) = (\alpha), \tag{2.1}$$

where (α) is the Cartier divisor for any nonzero rational global section α. The choice of α does not matter because if β is another nonzero rational global section, then

$$(\beta) = (\alpha) + (\beta/\alpha),$$

where β/α is a rational function, thus trivial in $\mathrm{Cl}(X)$.

To see c_1 is a group homomorphism, let L_1, L_2 be two line bundles over X not in the same isomorphism class. If α_1, α_2 are the corresponding rational global sections, then $\alpha_1 \otimes \alpha_2$ is the rational global section for $L_1 \otimes L_2$. The corresponding Cartier divisor is $(\alpha_1) + (\alpha_2)$. It follows that $c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$.

Assume X is locally factorial. It remains to show c_1 is bijective. Let α be any nonzero divisor, we can define L_α to be a coherent sheaf on X, whose sections on affine opens $U \subset X$ are given by the additive group of rational functions f such that $(f) + \alpha \geq 0$ on U. We claim that L_α is actually a line bundle. Since X is locally factorial, every irreducible divisor is defined locally by one function g, and so L_α locally consists of g^{-1} times the sheaf $\mathcal{O}_X$ of regular functions. Thus, L_α is locally isomorphic to $\mathcal{O}_X$, implying L_α is a line bundle. Now, for a rational function f, write (f) to be its corresponding divisor. $L_{(f)}$ is then trivial because multiplication by f gives an isomorphism from $L_{(f)}$ to $\mathcal{O}_X$. By construction, it is straightforward to see $c_1(L_\alpha) = \alpha$ and $c_1(L_{(f)}) = 0$. It follows that c_1 is surjective.

To see c_1 is injective, we suppose $c_1(L) = 0$ for a line bundle L over X. This implies that, for a rational global section $s \in \Gamma(X, L \otimes_{\mathcal{O}_X} \mathcal{R}_X)$, $(s) = (f)$ for some rational function f. Write $\overline{s} = s/f$. It is not hard to see $(\overline{s}) = 0$. So, $\overline{s}$ has no zeros or poles on any divisor on X. We claim that $\overline{s}$ has no zeros or poles on the whole X. In this case, $\overline{s}$ is a trivialization of L. Since X is factorial, the zeros and poles of rational functions on normal schemes occur only on the codimension-1 subvarieties by [Har13, II. Proposition 6.2 and 6.3A]. Thus, a rational function with no zeros or poles is in fact a unit. If we work locally on X, we can assume L is trivial. It follows that, L is now trivialized by a rational function with no zeros or poles, hence a unit. So, $[L] = 0$ as expected. ◻

2.3. **Chow groups and Chow rings.**

Chow groups can be viewed as some generalization of the divisor class groups. Before we give the precise definition, we need to generalize the notion of divisor groups in the following sense.

Definition 2.29. The group of **algebraic cycles** on X is the free abelian group generated by the set of subvarieties of X, denoted $Z(X)$. This is a graded free abelian group $Z(X) = \bigoplus_{i\geq 0} Z_i(X)$, where $Z_i(X)$ is the free abelian group generated by the set of subvarieties of X of dimension i, called **i-cycles**.

From the definition, if $\dim X = n$, then the set of $(n-1)$-cycles are exactly the set of divisors.

Definition 2.30. Two i-cycles D, D' on X are **rationally equivalent** if their difference $D - D'$ lies in the subgroup generated by all divisors (f) of all rational functions $f \in \kappa(y)$, where $\kappa(y) = \mathcal{O}_{X,y}/\mathfrak{m}_y$, and y is a generic point of some $(i+1)$-dimensional subvariety $Y \subset X$. Write $\sim_i$ to be such equivalence relation in $Z_i(X)$.

Remark 2.31. There is an alternative definition of rational equivalence that will be useful later. Consider the cartesian product of a k-scheme X and $\mathbb{P}^1$. Let $\pi_1 : X \times \mathbb{P}^1 \to X$ be the projection onto X. Let $Y \subset X \times \mathbb{P}^1$ be a $(i+1)$-dimensional subvariety such that the projection onto $\mathbb{P}^1$ through π_2. Then π_2 induces a field homomorphism

$$\pi_2^* : \kappa(\mathbb{P}^1) \to \kappa(Y),$$

sending a generic rational function f' on $\mathbb{P}^1$ to a rational function f on Y. Let $P \in \mathbb{P}^1$ be any point that is rational on the ground field, then $\pi_2^{-1}(P)$ is a subscheme of $X \times \{P\}$, viewed as a subscheme of X. It follows direction from definition that

$$(f) = \langle f^{-1}(0)\rangle - \langle f^{-1}(\infty)\rangle,$$

where the notion $\langle - \rangle$ specifies the divisors. In general, a cycle $\alpha \in Z_i(X)$ is rationally equivalent to 0, if and only if there are $(i+1)$-dimensional subvarieties $Y_1, , \cdots, Y_s \subset X \times \mathbb{P}^1$, such that

$$\alpha = \sum_{j=1}^{s} \langle f_j^{-1}(0)\rangle - \langle f_j^{-1}(\infty)\rangle,$$

where each f_j is some generic rational function on Y_j.

Definition 2.32. The **i-th Chow group** of X is defined to be

$$CH_i(X) = Z_i(X)/\sim_i .$$

If X is of dimension n, we write

$$CH^i(X) \coloneqq CH_{n-i}(X).$$

Corollary 2.33. *From the definition, $CH^i(X) \cong CH^i(X_{\text{red}})$, where X_{red} is the underlying reduced scheme of X. This is because the subvarieties of X_{red} are the same as ones of X.*

It is clear from the definition that $CH^1(X) = CH_{n-1}(X) = \mathrm{Cl}(X)$, where $n = \dim X$. Also, since X is the only n-dimensional subvariety of X and there is no higher dimensional subvariety, it follows that $CH^0(X) = CH_n(X) \cong \mathbb{Z}$. It follows that the Chow group of a 0-dimensional scheme is the free abelian group on the components.

Remark 2.34. One can view the Chow groups $CH_i(X)$ through the presentation given by the right short exact sequence

$$Z_i(\mathbb{P}^1_k \times X) \xrightarrow{\Psi} Z_i(X) \to CH_i(X) \to 0,$$

where $\Psi : \langle Y \rangle \mapsto \langle Y \cap (\{0\} \times X) \rangle - \langle Y \cap (\{\infty\} \times X) \rangle$ if the subvariety $Y \subset \mathbb{P}^1_k \times X$ is not contained in a fiber, and 0 if it does. Equivalently, there is also a short exact sequence

$$\coprod_{y \text{ generic in } \dim(i+1)\text{-subvariety}} \kappa(y) \to \coprod_{y \text{ generic in } \dim i\text{-subvariety}} \mathbb{Z} \to CH_i(X) \to 0. \tag{2.2}$$

These are basically a combination of Remark 2.31 and Definition 2.32.

In general, the computation of Chow groups requires the knowledge of all subvarieties of a scheme, and so are extremely hard. Still, there are some basic calculations we can accomplish. Before we do that, we first introduce the ring structure on Chow groups, as well as some computation techniques grown out of homology theory.

Similar to the ordinary cohomology, Chow groups in each degree packaged together admit a ring structure. To see this fact, the following lemma is needed.

Definition 2.35. Let Y, Z be two subvarieties of X. Y and Z **intersect transversely** at some point $p \in Y \cap Z$ if

$$T_pY \oplus T_pZ = T_pX,$$

or equivalently

$$\operatorname{codim}(T_pY \cap T_pZ) = \operatorname{codim}(T_pY) + \operatorname{codim}(T_pZ).$$

Y and Z are **generically transverse** if they intersect transversely at a general point of each component W of $Y \cap Z$.

Lemma 2.36 (The moving lemma, [DH16, Appendix A.])**.** *Let X be a smooth quasi-projective variety. Then for every $x_1, x_2 \in Z(X)$, there are generically transverse cycles $D_1, D_2 \in Z(X)$ such that $[D_1] = x_1$, $[D_2] = x_2$, and $[D_1 \cap D_2]$ is independent of the choice of D_1, D_2.*

Now, we are ready for the ring structure on Chow groups.

Theorem 2.37 ([DH16, Theorem 1.5])**.** *If X is a smooth quasi-projective variety, then there is a unique product structure on $CH^*(X)$ satisfying that for every generically transverse two subvarieties Y, Z of X,*

$$[Y] \cdot [Z] = [Y \cap Z].$$

Example 2.38. Let's calculate the first non-trivial example of Chow rings, which is the one of $X = \mathbb{A}^n_k$. We first compute its Chow groups. Let Y be any proper subvariety of X. Consider

$$Z = \{(tz, t) \in \mathbb{A}^n_k \times \mathbb{A}^1_k : z \in Y, t \in \mathbb{A}^1_k \backslash \{0\}\},$$

and let $\overline{Z} \subset \mathbb{A}^n_k \times \mathbb{P}^1_k$ be its projective closure. They are irreducible since Y is irreducible. From the construction, the fiber of $\overline{Z}$ at $t = 1$ is exactly Y. We can easily see that $Z = V(\{f(z/t) : f(z) = 0, \forall z \in Y\})$ as a vanishing locus. Since the origin of $\mathbb{A}^n_k$ does not lie in Y, there is some $g \in I(Y)$ such that $g(0) = c \neq 0$. The function $G(z, t) = g(z/t)$ is then an regular function for $\overline{Z}$, which has constant value c on $\infty \times \mathbb{A}^n_k$. So the fiber of $\overline{Z}$ at $t = \infty$ is empty, yielding that $Y \sim 0$. Thus,

$$CH^i(\mathbb{A}^n_k) = \begin{cases} \mathbb{Z} & , i = n, \\ 0 & , \text{else}. \end{cases}$$

Hence, the ring structure of $CH^*(\mathbb{A}^n_k)$ is trivial.

Theorem 2.39 (Excision)**.** *If $Z \subset X$ is a closed subscheme and $U = X \backslash Z$ is its complement, then the inclusion and restriction maps of cycles give a right exact sequence*

$$CH^*(Z) \to CH^*(X) \to CH^*(U) \to 0.$$

Moreover, if X is smooth, then $CH^*(X) \to CH^*(U)$ is a ring homomorphism.

Proof. Let $i : Z \to X$ be the inclusion and $j : X \to U$ be the restriction. By Remark 2.34, there is a commutative diagram with exact columns

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_i(Z \times \mathbb{P}^1) & \longrightarrow & Z_i(X \times \mathbb{P}^1) & \longrightarrow & Z_i(U \times \mathbb{P}^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_i(Z) & \longrightarrow & Z_i(X) & \longrightarrow & Z_i(U) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & CH_i(Z) & \xrightarrow{i_*} & CH_i(X) & \xrightarrow{j_*} & CH_i(U) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the rows are exact by construction (i.e. inclusions and restrictions on cycles). Take any cycle $\alpha \in Z_i X$. If $j_*\alpha \sim_i 0$, then $j_*\alpha = \sum(f_i)$ for some rational functions f_i on subvarieties of U. It follows that

$$j_*(\alpha - \sum(f_i)) = 0$$

on $Z_i(U)$ and hence there exists some $\beta \in Z_i(Y)$ such that $i_*(\beta) = \alpha - \sum(f_i)$. The exactness is then shown. □

Remark 2.40. Compare the excision theorem to Example 2.26. One can find that the excision is really a generalization of the conclusion in the previous example.

Theorem 2.41 (Mayer-Vietoris)**.** *If Y and Z are closed subschemes of X, then there is a right exact sequence of rings*

$$CH^*(Y \cap Z) \to CH^*(Y) \oplus CH^*(Z) \to CH^*(X) \to 0$$

Proof. Consider the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Z_i((Y \cap Z) \times \mathbb{P}^1) & \longrightarrow & Z_i(Y \times \mathbb{P}^1) \oplus Z_i(Z \times \mathbb{P}^1) & \longrightarrow & Z_i(X \times \mathbb{P}^1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Z_i(Y \cap Z) & \xrightarrow{i_*} & Z_i(Y) \oplus Z_i(Z) & \xrightarrow{j_*} & Z_i(X) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & CH_i((Y \cap Z)) & \longrightarrow & CH_i(Y) \oplus CH_i(Z) & \longrightarrow & CH_i(X) & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $i_*(\alpha) = (\alpha, -\alpha)$, and $j_*(\alpha_1, \alpha_2) = \alpha_1 + \alpha_2$ with $\alpha_1 \in Z_i Y$, $\alpha_2 \in Z_i Z$. The result directly follows from Theorem 2.39. □

Example 2.42. Let $X = \mathbb{P}^n_k$. By Theorem 2.39, taking $X = \mathbb{P}^n_k$, $Z = \mathbb{A}^n_k$, and $U = X \backslash Z = \mathbb{P}^{n-1}_k$, we get a short exact sequence

$$CH^i(\mathbb{A}^n_k) \to CH^i(\mathbb{P}^n_k) \to CH^i(\mathbb{P}^{n-1}_k) \to 0.$$

It follows from Example 2.38 that

$$CH^i(\mathbb{P}^n_k) = \begin{cases} \mathbb{Z} & , 0 \leq i \leq n, \\ 0 & , \text{else}. \end{cases}$$

In order to determine the ring structure, we need to examine the intersection of different subvarieties. Note that, the intersection of two linear subspaces of $\mathbb{P}^n_k$ is again linear subspace. It follows that

$$CH^*(\mathbb{P}^n_k) = \mathbb{Z}[u]/(u^{n+1}),$$

where u is the class of a hyperplane $\mathbb{P}^{n-1}_k \subset \mathbb{P}^n_k$, and u^i is the class of a linear subspace of codimension i for $0 \leq i \leq n$.

2.4. **Functoriality.** The overall aim of this chapter is to show that CH^* (resp. CH_*) is a well-defined contravariant (resp. covariant) functor from category of smooth k-schemes to the category of rings. We have already discussed the behavior of this assignment on the object-level in the last section. The section shoots for the behavior of this assignment on the morphism-level.

We divide our discussion into two cases: the contravariant one (corresponds CH^i) and the covariant one CH_i.

2.4.1. *Covariant case.* Let $f : X \to Y$ be a morphism of smooth k-schemes. It is called *proper* if it is separated and universally closed, i.e. for all $Y' \to Y$, the map from the pullback $X \times_Y Y' \to Y'$ is a closed map.

Let $M \subset X$ be any subvariety of Y of dimension d. Then $f(M)$ is a closed subvariety of Y of dimension d, and f induces a local homomorphism

$$f^* : \mathcal{O}_Y \mid_{f(M)} \to \mathcal{O}_X \mid_M .$$

At each generic point, f^* restricts to a homomorphism of residue fields $\kappa(f(M)) \to \kappa(M)$ such that $\kappa(M)$ is a finite field extension of $\kappa(f(M))$, because they are both finitely generated fields of the same transcendence degree $\dim M$ over the ground field. Geometrically, for any generic point $y \in f(M)$, its primage $x = f \mid_M^{-1} (y)$ is a finite scheme, and

$$[\kappa(M) : \kappa(f(M))] =: \deg(M/f(M))$$

measures the degree of x over y, or equivalently the number of covering branches. Now define

$$f_*\langle M \rangle \coloneqq \deg(M/f(M)) \cdot \langle f(M) \rangle. \tag{2.3}$$

This map extends linearly to a homomorphism

$$f_* : Z_i X \to Z_i Y. \tag{2.4}$$

Moreover, it follows that $(g \circ f)_* = g_* \circ f_*$ for a proper morphism $g : Y \to Z$ by the multiplicativity formula for extension degrees.

Proposition 2.43 ([Ful98, Proposition 1.4])**.** *If $f : X \to Y$ is a proper and surjective morphism of varieties, and r is a nonzero rational function on X, then*

$$f_*((r)) = \begin{cases} (\mathrm{Nm}(r)), & \dim Y = \dim X, \\ 0, & \dim Y < \dim X. \end{cases}$$

Here $(\mathrm{Nm}(r))$ is the divisor of $\mathrm{Nm}(r)$ (see Definition 2.22), and $\mathrm{Nm}(r)$ is the norm of r, i.e. the determinant of the $\kappa(y)^*$-linear endomorphism of $\kappa(x)^*$ given by multiplication by r, where $x \in X, y \in Y$ are generic points and $\kappa(y)$ is finite over $\kappa(x)$.

Proposition 2.44. *Let f be defined as the beginning. Then f induces a homomorphism for each i,*

$$f_* : CH_i(X) \to CH_i(Y),$$

sending $\langle M \rangle$ to $\deg(M/f(M)) \cdot \langle f(M) \rangle$.

Proof. By (2.3) and (2.4), it suffices to restrict to the case that $\langle M \rangle$ is a divisor in $Z_i X$. Let $\alpha = \langle M \rangle \in Z_i X$ that is rationally equivalent to 0. So we can write $\alpha = (r)$ for some nonzero rational function f on some subvariety W of X. Replace X by W in the setting, and Y by $f(W)$. We may assume $f(W)$ is a variety and f is surjective. By Proposition 2.43, $f_*(\alpha) = f_*((r))$ is again rationally equivalent to a function. Thus, f_* is well-defined on the level of Chow groups. The proposition follows. ☐

2.4.2. *Contravariant case.* Let $f : X \to Y$ be a morphism of smooth k-schemes. It is called *flat* if it corresponds to a flat morphism of rings on each affine open subset. The assumption of flatness ensures that, when Y is connected, all nonempty fibers of f have the same dimension. Thus, we may assume that f is *of relative dimension* d, meaning that all fibers are equidimensional of dimension d.

Let $M \subset Y$ be any subvariety of Y of dimension d.

Proposition 2.45. *Let f be defined as above. Then f induces a homomorphism for each i,*

$$f^* : CH^i(Y) \to CH^{i+d}(X),$$

sending $\langle M \rangle$ to $\langle f^{-1}(M) \rangle$.

Note that $f^{-1}(M)$ is of pure dimension $i + d$. The assignment

$$f^* : \langle M \rangle \mapsto \langle f^{-1}(M) \rangle$$

can be easily extended to a homomorphism

$$f^* : Z_i(Y) \to Z_{i+d}(X) \tag{2.5}$$

by linearity. To prove the proposition, we need to study the interaction between f^* and rational equivalence. Before that, we need the following propositions.

Proposition 2.46 ([Ful98, Lemma 1.7.1])**.** *If $f : X \to Y$ is flat, then for any subscheme Z of Y,*

$$f^*(\langle Z \rangle) = \left\langle f^{-1}(Z) \right\rangle.$$

Proposition 2.47 ([Ful98, Proposition 1.7])**.** *Consider the following pullback diagram of smooth k-schemes:*

$$\begin{tikzcd} X' \arrow[r, "\phi'"] \arrow[d, "i'"'] & X \arrow[d, "i"] \\ Y' \arrow[r, "\phi"'] & Y \end{tikzcd}$$

where ϕ is flat, and i is proper. Then ϕ' is flat, i' is proper, and any i-cycle $\alpha \in Z_i(X)$, we have, in $Z_i(Y')$,

$$i'_*(\phi')^*(\alpha) = \phi^* i_*(\alpha)$$

Definition 2.48. Let X be a scheme with irreducible components $X_1, \cdots, X_s$. The **geometric multiplicities** m_i of X_i in X is defined to be

$$m_i = \ell(\mathcal{O}_{X,\xi_i}),$$

where $\xi_i \in X_i$ is a generic point.

Proposition 2.49 ([Ful98, Lemma 1.7.2])**.** *Let X be a n-dimensional scheme with irreducible components $X_1, \cdots, X_s$ and geometric multiplicities $m_1, \cdots, m_s$. Let D be an effective Cartier divisor on X (see Remark 2.20), and $D_i = D \cap X_i$ be the restriction of D to X_i. Then in $Z_{n-1}X$,*

$$D = \sum_{i=1}^{s} m_i \cdot D_i.$$

Actually, Proposition 2.49 also holds if the Cartier divisor D is not effective.

Proof of Proposition 2.45. By (2.5), it suffices to restrict to the case that $\langle M \rangle$ is a divisor in $Z_i Y$. Let $\alpha = \langle M \rangle \in Z_i(Y)$ and $\alpha \sim_i 0$. We claim that $f^*\alpha \sim_{i+d} 0$ in $Z_{i+d}(X)$.

By Remark 2.31, we may assume $\alpha = \langle g^{-1}(0) \rangle - \langle g^{-1}(\infty) \rangle$ for some generic rational function g on $Y_0 \subset Y \times \mathbb{P}^1$. Let $\pi_2 : Y_0 \subset Y \times \mathbb{P}^1 \to \mathbb{P}^1$ be the projection. It is clear π_2 is flat. Denote $W = (f \times 1)^{-1}(Y_0)$. It is a closed subscheme of $X \times \mathbb{P}^1$. Let $h : W \to \mathbb{P}^1$ be the projection. Write $p : X \times \mathbb{P}^1 \to X$ and $q : Y \times \mathbb{P}^1 \to Y$ to be the projections. Then

$$\begin{aligned} f^*(\alpha) &= f^* q_*(\langle \pi_2^{-1}(0) \rangle - \langle \pi_2^{-1}(\infty) \rangle) \\ &= p_*(f \times 1)^*(\langle \pi_2^{-1}(0) \rangle - \langle \pi_2^{-1}(\infty) \rangle) && \text{(by Proposition 2.47)} \\ &= p_*(\langle h^{-1}(0) \rangle - \langle h^{-1}(\infty) \rangle) && \text{(by Proposition 2.46)} \end{aligned}$$

Let $W_1, \cdots, W_s$ be the irreducible components of W with geometric multiplicities $m_1, \cdots, m_s$, and $h_i = h \mid_{W_i}$ for each i. It follows that $\langle W \rangle = \sum_{i=1}^s m_i \cdot \langle W_i \rangle$. From the fact

$$\langle h_i^{-1}(0) \rangle - \langle h_i^{-1}(\infty) \rangle = (h_i),$$

it suffices to show that

$$\langle h^{-1}(P) \rangle = \sum_{i=1}^{s} \langle h_i^{-1}(P) \rangle$$

for $P = 0, \infty$. This follows from Proposition 2.49. $\square$

2.5. **Borel-Moore homology.** Borel–Moore homology, also known as locally finite homology, is an important tool in constructing Poincaré duality for non-compact manifolds. Let X be a locally compact manifold, and R be a nice commutative ring.

Definition 2.50. The **Borel-Moore homology** $H_i^{BM}(X;R) = \varprojlim H_i(X, X\backslash K)$, where K runs over all compact subsets of X.

Remark 2.51. There is another definition of Borel-Moore homology via the locally finite chain complex. Namely, it is the homology of

$$\cdots \to C_{n+1}^{BM}(X;R) \xrightarrow{d_n} C_n^{BM}(X;R) \xrightarrow{d_{n-1}} C_{n-1}^{BM}(X;R) \to \cdots ,$$

where $C_n^{BM}(X;R) = \{\sum r_\sigma \cdot \sigma : r_\sigma \in R\}$ is the abelian group of formal (possibly infinite) summation of singular chains $\sigma : \Delta^n \to X$, such that for each compact subset $K \subset X$, we have $a_\sigma \neq 0$ for only finitely many σ whose image meets K. The boundary map d coincides with the usual definition of boundary map. It is easy to show $d^2 = 0$, and the resulting homology coincides with the one in Definition 2.50. A detailed discussion lies beyond the scope of this paper and will therefore be omitted.

Proposition 2.52. *If X is a compact space, then $H_i^{BM}(X;R) = H_i(X;R)$.*

Proof. Taking $K = X$ in Definition 2.50 yields the desired result. □

Example 2.53. If $X = \mathbb{R}^n$, then $H_i^{BM}(X;R) = R$ for $i = n$, and 0 elsewhere. This can be reduced to the case when $n = 0$. One observes that any point p admits a locally finite chain $\sum_{k\in\mathbb{N}}[p+k, p+k+1]$ without boundary, yielding the desired result.

As one would expect, Borel-Moore homology satisfies a lot of properties that hold for ordinary homology theory: long exact sequence, excision, Mayer-Vietoris, grading, etc. The discussion of these properties are beyond our scope. Interested readers are encouraged to consult [AC60] for more details. Nevertheless, the relation between Chow groups and Borel-Moore homology is worth mentioning separately.

Let X be a scheme over $\mathbb{C}$. X might not be proper. In this case, $X(\mathbb{C})$ is not necessary compact in the classical topology. So closed subvarieties of X also need not be compact. Now, it is not appropriate to treat $CH_*(X)$ as an ordinary cohomology theory anymore, but rather a Borel-Moore homology theory.

Proposition 2.54 ([Ful98, Proposition 19.1.1])**.** *There is a natural group homomorphism, called **cycle map**, from Chow groups to Borel-Moore homology*

$$cl_i : CH_i(X) \to H_{2i}^{BM}(X;\mathbb{Z}).$$

Chow groups appear also as a component of Voevodsky's theory of motivic cohomology, which is deeply connected to the Milnor K-theory, to be introduced in §3.1. For the purpose of our paper, we will only focus on the latter topic.

3. Milnor-Rost cycle modules

3.1. **Milnor K-theory.** Fix a field k. Consider the tensor algebra $T(k^*) = \bigoplus_{m\geq 0}(k^*)^{\otimes m}$, and write $l(x)$ to be the degree element of degree 1 in $T(k^*)$ corresponding to $x \in k^*$.

Definition 3.1. The **graded Milnor K-ring** is $\mathrm{K}_*^M(k) \coloneqq T(k^*)/\langle l(x)\otimes l(1-x)\rangle$, where $x \neq 0,1$. For each $n \geq 0$, the n**-th Milnor K-group** of k, denoted $\mathrm{K}_n^M(k)$, is defined to be the free abelian group generated by the elements

$$l(x_1)\otimes l(x_2)\otimes\cdots\otimes l(x_n),$$

where the elements $x_1,\cdots,x_n \in k^*$ subject to

(a) multilinear relation:

$$\begin{aligned} l(x_1)\otimes\cdots l(x_k x_{k+1})\otimes\cdots\otimes l(x_n) &= l(x_1)\otimes\cdots\otimes l(x_k)\otimes\cdots\otimes l(x_n) \\ &+ l(x_1)\otimes\cdots\otimes l(x_{k+1})\otimes\cdots\otimes l(x_n) \end{aligned}$$

, and

(b) if $x_i + x_{i+1} = 1$ for some i, then $l(x_1)\otimes\cdots\otimes l(x_n) = 0$.

We use the notation $\{x_1,x_2,\cdots,x_n\}$ to denote $l(x_1)\otimes l(x_2)\otimes\cdots\otimes l(x_n)$. Now two conditions in Definition 3.1 can be rephrased as follows:

(a) $\{x_1,\cdots,x_kx_{k+1},\cdots,x_n\} = \{x_1,\cdots,x_k,\cdots,x_n\} + \{x_1,\cdots,x_{k+1},\cdots,x_n\}$;
(b) if $x_i + x_{i+1} = 1$ for some i, then $\{x_1,\cdots,x_n\} = 0$.

In fact, the ring structure of $\mathrm{K}_*^M(k)$ can be defined by such symbols, where the product operation is given by

$$\{x_1,x_2,\cdots,x_n\}\cdot\{y_1,y_2,\cdots,y_m\} = \{x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_m\}. \tag{3.1}$$

Example 3.2. When $n = 0,1$, it is not hard to see $\mathrm{K}_0^M(k) = \mathbb{Z}$ and $\mathrm{K}_1^M(k) = k^*$. In particular, for $\mathrm{K}_1^M(k)$, one has $\{xy\} = \{x\} + \{y\}$ for all $x, y \in k^*$.

Although the discussion of the algebraic K-theory is beyond our scope, we would like to introduce one of the comparison between the Milnor K-theory and the algebraic K-theory as the following theorem.

Theorem 3.3 (Matsumoto, [Wei13, Theorem III.6.1])**.** *The second algebraic K-group of fields coincides the second Milnor K-group of fields:* $\mathrm{K}_2(k) \cong \mathrm{K}_2^M(k)$.

Example 3.4. Let $\sigma \in S_n$ be a permutation. Let $x_1,\cdots,x_n \in k^*$. Then

$$\{x_{\sigma(1)},x_{\sigma(2)},\cdots,x_{\sigma(n)}\} = \mathrm{sgn}(\sigma)\{x_1,x_2,\cdots,x_n\}.$$

This follows from the condition (b) in Definition 3.1, where we note that $\{x_i,x_{i+1}\} + \{x_{i+1},x_i\} = 0$ for each i (by Theorem 3.3 and properties of Steinberg symbols). Moreover, if $x_i + x_j = 1$ or 0 for some $i \neq j$, then $\{x_1,\cdots,x_n\} = 0$, generalizing the condition (b) in Definition 3.1.

Example 3.5. Let $k = \mathbb{F}_p$. We have $\mathrm{K}_n^M(k) = 0$ for each $k \geq 2$. This is because by Theorem 3.3, $\mathrm{K}_2^M(k) \cong \mathrm{K}_2(k) = 0$, and higher Milnor K-groups of k contains elements like $l(x)\otimes l(y) = 0$ for some $x, y \in k^*$. This is an example of algebraic K-theory contains more information than then Milnor K-theory: the former one is in fact nonzero for odd n. See [Qui72].

We can talk about the homomorphism between Milnor K-groups. Consider a field homomorphism $f : k \to \ell$. f induces the *restriction* map on graded rings

$$\mathrm{K}_*^M(f) : \mathrm{K}_*^M(k) \to \mathrm{K}_*^M(\ell) \tag{3.2}$$

by taking $\{x_1,\cdots,x_n\}$ in k to $\{f(x_1),\cdots,f(x_n)\}$ in l. Write $r_{\ell/k} \coloneqq \mathrm{K}_*^M(f)$. If $g : \ell \to \jmath$, then the induced maps on Milnor rings satisfy $r_{\jmath/k} = r_{\jmath/\ell}\circ r_{\ell/k}$. It follows that $\mathrm{K}_*^M(-)$ is a functor from the category of fields to the category of graded rings.

Proposition 3.6. *Let ℓ/k be a quadratic field extension (i.e. of finite degree 2). We have, for every $n \geq 1$,*
$$\mathrm{K}_n^M(\ell) = r_{\ell/k}(\mathrm{K}_{n-1}^M(k)) \cdot \mathrm{K}_1^M(k).$$

Proof. Suffice to prove it for $n = 2$, and the general case follows from the chain relation $r_{\jmath/k} = r_{\jmath/\ell} \circ r_{\ell/k}$ together with the induction.

Now, for any $\{x, y\} \in \mathrm{K}_2^M(\ell)$, $x, y \in \ell \backslash k$, if $x = cy$ for some $c \in k^*$, then
$$\{x, y\} + \{x, c\} = 0,$$
yielding that $\{c, x\} = \{x, y\}$. Note that this implies that $\{x, y\} = \{c\} \cdot \{y\} \in r_{\ell/k}(\mathrm{K}_{n-1}^M(k)) \cdot \mathrm{K}_1^M(k)$. If $x = cy$ does not hold, then there exists $a, b \in k^*$ such that $ax + by = 1$, yielding that
$$\begin{aligned} 0 = \{ax, by\} &= \{a, by\} + \{x, by\} \\ &= \{a, by\} + \{x, y\} + \{x, b\} \end{aligned}$$
So $\{x, y\} = \{b, x\} - \{a, by\} = \{b\} \cdot \{x\} - \{a\} \cdot \{by\} \in r_{\ell/k}(\mathrm{K}_{n-1}^M(k)) \cdot \mathrm{K}_1^M(k)$. $\square$

We now describe the Milnor's theorem. Before we do that, we need to review some knowledge in number theory.

Definition 3.7. For k a field, a **valuation** on k is a function $\nu : k \to \mathbb{R} \cup \{\infty\}$ such that

(a) $\nu(x) = 0$ if and only if $x = 0$,
(b) $\nu(xy) = \nu(x) + \nu(y)$,
(c) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$.

The **valuation ring** of k is $\mathcal{O}_k \coloneqq \{x \in k : \nu(x) \geq 0\}$.

Definition 3.8. If k is a field with a valuation ν, k is a **discrete valuation field** (**DVF**) if $\nu(k^*) \subset \mathbb{R}$ is a discrete subgroup of $\mathbb{R}$, i.e. $\nu(k^*)$ is infinite cyclic.

Definition 3.9. Let k be a DVF with a valuation ν. A **uniformizer** $\pi \in k$ is an element such that $\nu(\pi) > 0$, and $\nu(\pi)$ generates $\nu(k^*)$.

Example 3.10. If $k = \mathbb{Q}_p$, the p-adic numbers, then $\nu_p(x) = -\log_p |x|_p$ is an example of valuation. Actually, $(\mathbb{Q}_p, \nu_p)$ is a DVF. Now p is a uniformizer of k.

Example 3.11. Let K be a DVF, $k = \mathcal{O}_K/\mathfrak{m}_K$, where $\mathfrak{m}_K$ is the unique maximal ideal of $\mathcal{O}_K$. Then $k((x)) = \{\sum_{i=n}^{\infty} a_i x^i : a_i \in k, \ n \in \mathbb{Z}\}$, i.e. the field of formal Laurent series over k, has a valuation
$$\nu\left(\sum_{i=n}^{\infty} a_i x^i\right) = \min\{i : a_i \neq 0\}.$$
A choice of uniformizer for $k((x))$ is x.

Definition 3.12. A **local field** is a complete DVF with finite residue field.

Definition 3.13. Let R be a ring. It is a **discrete valuation ring** (**DVR**) if it is a PID with a unique prime element up to units.

Example 3.14. Let k be a field with a valuation. Then $\mathcal{O}_K$ is a DVR.

Proposition 3.15. *R is a DVR if and only if $R \cong \mathcal{O}_K$ for some DVF K.*

Proof. It suffices to prove the "if" part. Let π be the prime in R. For any nonzero $x \in R$, there exists a unique unit u and $n \geq 0$, and $x = \pi^n u$. Let ν be defined as

$$\nu(x) = \begin{cases} n & , x \neq 0; \\ \infty & , x = 0. \end{cases}$$

Observe that $R = \mathcal{O}_K$ for $K = R[1/\pi]$. Thus,

$$\nu(\pi^n u) = n \Leftrightarrow \pi^n u \in R.$$

□

Let ℓ/k be an extension of local fields with $\mathfrak{m}_k \subset \mathfrak{m}_\ell$, $\mathcal{O}_k \subset \mathcal{O}_\ell$. Then we have an injection of residue fields $\kappa(k) \hookrightarrow \kappa(\ell)$.

Definition 3.16. For ℓ/k of the above setting, we have the

(a) **inertia degree**, which is $f_{\ell/k} = [\kappa(\ell) : \kappa(k)]$;
(b) **ramification index**, which is $e_{\ell/k} = \nu_\ell(\pi_k)$, where ν_ℓ is the (normalized) valuation of ℓ, and π_k is the uniformizer of k.

Theorem 3.17. *If ℓ/k is finite, then*

$$[\ell : k] = e_{\ell/k} \cdot f_{\ell/k}.$$

Proof. See [Mil20, Theorem 3.34]. □

Let X be a variety over k (integral k-scheme), and $x \in X$ be a regular point of codimension 1. The local ring $\mathcal{O}_{X,x}$ is then a DVR (actually a field) with valuation ν measuring the orders of vanishing of regular functions at x, whose residue field is $\kappa(x)$ as in Chapter 2. There is a *residue homomorphism* for each $n \geq 0$

$$\partial_{x,\nu} : \mathrm{K}^M_{n+1}(F(X)) \to \mathrm{K}^M_n(\kappa(x)), \tag{3.3}$$

where $F(X) = \mathrm{Frac}(\mathcal{O}_{X,x})$. This is given by, for any $f_0, f_1, \cdots, f_n \in F(X)^*$ satisfying $\nu(f_i) = 0$ for all $i = 1, 2, \cdots, n$, then

$$\partial_{x,\nu}(\{f_0, f_1, \cdots, f_n\}) = \nu(f_0) \cdot \{\bar{f}_1, \bar{f}_2, \cdots, \bar{f}_n\},$$

where $\bar{f}_i \in \kappa(x)$ is the corresponding image in the residue field $\kappa(x)$. We will omit one of x or ν in the notion $\partial_{x,\nu}$ if we are not specifying any element or valuation.

Example 3.18. There is a natural map from $\mathrm{K}^M_n(k) \to \mathrm{K}^M_n(k(x))$ for each n, where x is a variable. Define the leading coefficient map $\lambda(f) = \frac{a_0}{b_0}$ for $f = \frac{a_0 x^n + a_1 x^{n-1} + \cdots + a_n}{b_0 x^m + b_1 x^{m-1} + \cdots + b_m} \in k(x)^*$. Then λ induces a map on Milnor K-groups $\lambda : \mathrm{K}^M_n(k(x)) \to \mathrm{K}^M_n(k)$, namely

$$\lambda : \{f_1, \cdots, f_n\} \mapsto \{\lambda(f_1), \cdots, \lambda(f_n)\}.$$

It follows that λ is a well-defined map in the sense of Definition 3.1. Therefore, $\mathrm{K}^M_n(k)$ is a direct summand of $\mathrm{K}^M_n(k(t))$.

We are ready to introduce the Milnor's theorem. Taking $X = \mathbb{A}^1_k$ into (3.3) and letting x be a generic point of X, we have $F(X) = \mathrm{Frac}(\mathcal{O}_{X,x}) = k(t)$, where t is some variable. Set $n \geq 0$. Write L_d to be the subgroup of $\mathrm{K}^M_{n+1}(k(t))$ generated by $\{f_1, \cdots, f_n\}$ with $\deg f_j \leq d$ for all j. Let $h \in F(X) = k(t)$ be a polynomial of degree d and $k' = k[t]/h$. Denote $\bar{f}$ for the image of $f \in k[t]$ in k' (which is a polynomial of degree less than d).

Lemma 3.19. *There is a unique homomorphism* $\alpha_h : \mathrm{K}^M_{n-1}(k') \to L_d/L_{d-1}$ *sending* $\{\bar{f}_2, \cdots, \bar{f}_n\}$ *to* $\{h, f_2, \cdots, f_n\} \mod L_{d-1}$*, such that it, as well as* ∂_x*, induce an isomorphism between* L_d/L_{d-1} *and* $\oplus_h \mathrm{K}^M_{n-1}(k')$ *as* h *ranges over all monic irreducible polynomials of degree* d *in* $k[t]$*.*

Proof. To see α_h is a homomorphism, it suffices to prove the multilinearity in each factor, which suffices to show it is linear for $\bar{f}_2$. Assume $\bar{f}_2 = \bar{f}_2{}' \bar{f}_2{}''$. If $f_2 \neq f_2' f_2''$, then there is some nonzero polynomial f of degree less than d, such that $f_2 = f_2' f_2'' + fh$. Note that $fh/f_2 = 1 - f_2' f_2''/f_2$. It follows that $\{fh/f_2, f_2' f_2''/f_2\} = 0$. Multiplying $\{f_3, \cdots, f_n\}$ yields

$$\{h, f_2' f_2''/f_2, f_3, \cdots, f_n\} \equiv 0 \mod L_{d-1}.$$

Note that if $\bar{f}_i + \overline{f_{i+1}} = 1$ in k', then $f_i + f_{i+1} = 0$ in k. Thus, α_h factors through $\mathrm{K}^M_{n-1}(k')$.

Now, since h cannot divide any polynomial of degree less than d, ∂_x vanishes on L_{d-1} and it induces maps $\overline{\partial}_x : L_d/L_{d-1} \to \mathrm{K}^M_{n-1}(k')$. By observation, we see $\coprod_x \overline{\partial}_x \circ \oplus_h \alpha_h$ is the identity on $\oplus_h \mathrm{K}^M_{n-1}(k')$. It suffices to show $\oplus_h \alpha_h$ is surjective onto L_d/L_{d-1}. Observe that L_d is generated by L_{d-1} together with the collection of $\{h, f_2, f_3, \cdots, f_n\}$, where each $\deg f_i < d$. The latter is actually in the image of α_h by construction, thus the surjectivity is proved. $\square$

Theorem 3.20 (Milnor)**.** *For every field* k *and* $n \geq 0$*, we have a split exact sequence*

$$0 \to \mathrm{K}^M_{n+1}(k) \xrightarrow{r_{k(x)/k}} \mathrm{K}^M_{n+1}(k(t)) \xrightarrow{\coprod_{x\in X} \partial_x} \coprod_{x \in X} \mathrm{K}^M_n(\kappa(x)) \to 0.$$

Proof. Induction on the degree d. The base case is done by Example 3.18, and noticing that L_0 is a direct summand isomorphic to $\mathrm{K}^M_{n+1}(k)$. The inductive step then follows from Lemma 3.19. $\square$

We will end this section by introducing the norm map. Let ℓ/k be a finite field extension, generated by the element a. There is a natural map $\ell^* \to k^*$ by restriction, which can be viewed as $\mathrm{K}^M_1(\ell) \to \mathrm{K}^M_1(k)$. This is an example of the norm map. In general,

Definition 3.21. There is a **norm map** $c_{\ell/k} : \mathrm{K}^M_n(\ell) \to \mathrm{K}^M_n(k)$ for each $n \geq 0$ satisfying that

(a) if ℓ/k is simple, generated by $a \in \ell$, then for each element $\alpha \in \mathrm{K}^M_n(\ell) = \mathrm{K}^M_n(k(a))$, there is a element $\beta \in \mathrm{K}^M_{n+1}(k(t))$ such that

$$\partial_x(\beta) = \begin{cases} \alpha & , x = a, \\ 0 & , \text{else}. \end{cases}$$

This is valid because we can identify ℓ with the residue field $k(a)$ of a closed point $a \in \mathbb{A}^1_k$, and the existence of β is generated by the Milnor theorem 3.20. Now set

$$c_{\ell/k}(\alpha) = -\partial_{\nu_\infty}(\beta),$$

where ν_∞ is the valuation on $F(\mathbb{A}^1_k) = k(t)$ with t^{-1} being a generator of the associated unique maximal ideal $\mathfrak{m}_{\nu_\infty}$, such that

$$\nu_\infty(f) = -\deg(f)$$

for any $f \in k(t)^*$.

(b) In general, if ℓ/k is finite such that we have a nested family of field extensions
$$k = k_0 \subset k_1 \subset k_2 \subset \cdots \subset k_N = \ell,$$
where each k_{i+1}/k_i is simple, then
$$c_{\ell/k} = c_{k_1/k_0} \circ c_{k_2/k_1} \circ \cdots \circ c_{k_N/k_{N-1}}.$$

Remark 3.22. In the part (a) of Definition 3.21, we have specified a valuation ν_∞. In fact, every other discrete valuation ν that is trivial on k, determines and is determined by a monic irreducible polynomial $\pi_\nu \in k[t]$. This is true because for every monic irreducible polynomial $p(t) \in k[t]$, one can define a discrete valuation ν by $\nu_p(f) = n$ for which $f = p(t)^n \cdot u$ for some unit u. For the converse, given a valuation $\nu \neq \nu_\infty$, the associated local ring $k[\nu] = \{f \in k[t] : \nu(f) \geq 0\}$ has a unique maximal ideal $\mathfrak{m}_\nu$. $\mathfrak{m}_\nu = (\pi_\nu)$ for some monic irreducible polynomial π_ν since $k[t]$ is a PID. Thus, we know there is an one-to-one correspondence between $\nu \neq \nu_\infty$ and π_ν.

Moreover, π_ν generates the corresponding maximal ideal $\mathfrak{m}_\nu$. The corresponding residue field is $k[t]/(\pi_\nu)$.

Proposition 3.23 (Projection formula)**.** *Let ℓ/k be simple, generated by $a \in \ell$. For each $\alpha \in \mathrm{K}_*^M(\ell)$ and $\beta \in \mathrm{K}_*^M(k)$, we have*
$$c_{\ell/k}(r_{\ell/k}(\beta) \cdot \alpha) = \beta \cdot c_{\ell/k}(\alpha).$$

Proof. Note $r_{\ell/k}(\beta) \cdot \alpha = \{i(\beta), \alpha\} = \{\beta, \alpha\}$, where $i : k \to \ell$ is the canonical inclusion. Also $\beta \cdot c_{\ell/k}(\alpha) = \{\beta, c_{\ell/k}(\alpha)\}$. By Milnor's theorem 3.20, treating $\mathrm{K}_*^M(k(t))$ and $\mathrm{K}_*^M(\kappa(x))$ as graded modules over $\mathrm{K}_*^M(k)$, we can treat $c_{\ell/k}$ as a morphism between $\mathrm{K}_*^M(k)$-modules of shift degree 0. The result then follows from counting the degrees and checking where the elements live. $\square$

Corollary 3.24. *If $[\ell : k] = d$, then the composition*
$$\mathrm{K}_*^M(k) \xrightarrow{r_{\ell/k}} \mathrm{K}_*^M(\ell) \xrightarrow{c_{\ell/k}} \mathrm{K}_*^M(k)$$
is the multiplication by d.

Proof. Taking $\alpha = 1 \in \mathrm{K}_0^M(\ell) \subset \mathrm{K}_*^M(\ell)$ into Proposition 3.23 yields
$$c_{\ell/k} \circ r_{\ell/k}(\beta) = \beta \cdot c_{\ell/k}(1).$$
By (a) of Definition 3.21, note that $k[t]$ is a UFD with quotient field $k(t)$, and so every $f \in k(t)^*$ satisfies
$$\begin{aligned} f &= C \cdot \prod_{p \text{ monic irreducible}} p(t)^{\nu(f)} \\ &= C \cdot \prod_{\nu \neq \nu_\infty} \pi_\nu^{\nu(f)} \qquad \text{(by Remark 3.22)} \end{aligned}$$
where C is the leading coefficient of f. Write $k[\nu] = \{f \in k[t] : \nu(f) \geq 0\}$ to be the associated local ring, and k_ν to be the corresponding residue field. Hence,
$$\sum_{\nu \neq \nu_\infty} [k_\nu : k] \cdot \nu(f) = \sum_{\nu \neq \nu_\infty} \deg(\pi_\nu) \cdot \nu(f) = \deg(f) = -\nu_\infty(f).$$
It follows that, by Definition 3.21,
$$c_{\ell/k}(1) = -\partial_{\nu_\infty}(f) = -\nu_\infty(f) = \deg(f) = [\ell : k] = d.$$

for some generic $f \in \mathrm{K}_1^M(k(t)) = k(t)^*$. $\square$

The well-definedness of the norm map is known as the Kato theorem. The proof is rather long and technical, and beyond our goal of the paper. We refer the interested readers to [Hes05, Theorem 3].

Theorem 3.25 (Kato)**.** *The norm map $c_{\ell/k}$ is well-defined, i.e. independent of the choice of generators of extensions.*

3.2. **Rost cycle premodules.** In §1 and (2.2) of Remark 2.34, we see the Chow groups $CH_i(X)$ for each $i \geq 0$ satisfy

$$\coprod_{y \in X_{(i+1)}} \kappa(y) \to \coprod_{y \in X_{(i)}} \mathbb{Z} \to CH_i(X) \to 0, \tag{3.4}$$

where $X_{(i)}$ is the dimension i subvariety of X. Together with the definition of the Milnor K-theory and Example 3.2, we see (3.4) can be reformulated into

$$\coprod_{y \in X_{(i+1)}} \mathrm{K}_1^M(\kappa(y)) \xrightarrow{\partial_y} \coprod_{y \in X_{(i)}} \mathrm{K}_0^M(\kappa(y)) \to CH_i(X) \to 0. \tag{3.5}$$

In general, this complex can be prolonged into a chain complex, which is known as the cycle complex

$$\cdots \to \coprod_{y \in X_{(i+1)}} M(\kappa(y)) \xrightarrow{d} \coprod_{y \in X_{(i)}} M(\kappa(y)) \xrightarrow{d} \coprod_{y \in X_{(i-1)}} M(\kappa(y)) \to \cdots$$

where $M(-)$ is a mysterious functor, known as the *cycle module*. In order to make sense of this concept, we first need to define the *cycle premodules* [Ros96]. Before we give the long definition, we need to make the following conventions.

Conventions 3.26. Let X be a k-scheme. We say F is a *field* over X, if F is a field itself, together with a morphism $\operatorname{Spec} F \to X$ such that F is a finite extension over k. A *valuation* on $F \neq k$ (over X) is a discrete valuation ν on F together with a morphism $\operatorname{Spec}(\mathcal{O}_\nu) \to X$ (here $\mathcal{O}_\nu$ is the corresponding valuation ring, and $\kappa(\nu)$ is its residue field) such that ν is of *geometric type* over k. The latter means:

(a) $k \subset \mathcal{O}_\nu$,
(b) $\kappa(\nu)$ is a finite extension over k and satisfies $\operatorname{tr.deg}_k(\kappa(\nu)) + 1 = \operatorname{tr.deg}_k(F)$.

A morphism between fields F_1, F_2 over X is a field homomorphism $F_1 \to F_2$ such that there is a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} F_1 \arrow[rr] \arrow[dr] & & \operatorname{Spec} F_2 \arrow[dl] \\ & X & \end{tikzcd}$$

such that each downwards arrow corresponds to a finite extension over k.

Notation. Write $\mathfrak{F}_X$ to be the category of fields over X as defined in the above convention (3.26), and write Ab to be the category of abelian groups.

Definition 3.27 (Rost)**.** A **cycle premodule** is a functor $M : \mathfrak{F}_X \to \mathsf{Ab}$ together with a $\mathbb{Z}/2$-grading $M = M_0 \oplus M_1$ or a $\mathbb{Z}$-grading $M = \coprod_n M_n$ and with the following data D1-D4 and rules R1a-R3e:

D1 For each $\varphi : F \to E$, there is $\varphi_* : M(F) \to M(E)$ of degree 0.
D2 For each finite extension $\varphi : F \to E$, there is $\varphi^* : M(E) \to M(F)$ of degree 0.

D3 For each F, the group $M(F)$ is equipped with a left $\mathrm{K}_*^M(F)$-module structure denoted by $x \cdot \rho$ for $x \in \mathrm{K}_*^M(F)$ and $\rho \in M(F)$. The product respects the grading $\mathrm{K}_n^M(F) \cdot M_m(F) \subset M_{n+m}(F)$.

D4 For each valuation ν on F, there is a map $\partial_\nu : M(F) \to M(\kappa(\nu))$ of degree -1.

R1a For each $\varphi : F \to E$, $\psi : E \to L$, one has $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$.

R1b For each finite extension $\varphi : F \to E$, $\psi : E \to L$, one has $(\psi \circ \varphi)^* = \varphi^* \circ \psi^*$.

R1c Let $\varphi : F \to E$, $\psi : F \to L$ with φ finite. Let $R = L \otimes_F E$. For each $p \in \operatorname{Spec} R$, and let $\varphi_p : L \to R/p$, $\psi_p : E \to R/p$ be the natural maps. Moreover, let l_p be the length of the localized ring $R_{(p)}$. Then

$$\psi_* \circ \varphi^* = \sum_p l_p \cdot (\varphi_p)^* \circ (\psi_p)_*.$$

R2 Consider the map $\varphi : F \to E$, and $x \in \mathrm{K}_*^M(F)$, $y \in \mathrm{K}_*^M(E)$, $\rho \in M(F)$, $\mu \in M(E)$. One has the following formulas (R2a)-(R2c):

R2a $\varphi_*(x \cdot \rho) = \varphi_*(x) \cdot \varphi_*(\rho)$.

R2b If φ is furthermore finite, then $\varphi^*(\varphi_*(x) \cdot \mu) = x \cdot \varphi^*(\mu)$.

R2c If φ is furthermore finite, then $\varphi^*(y \cdot \varphi_*(\rho)) = \varphi^*(y) \cdot \rho$.

R3 For a prime π of ν on F, we set

$$s_\nu^\pi : M(F) \to M(\kappa(\nu))$$

such that $s_\nu^\pi(\rho) = \partial_\nu(\{-\pi\} \cdot \rho)$.

R3a Let $\varphi : E \to F$ and let ν be a valuation on F which restricts to a nontrivial valuation ω on E with ramification index e. Let $\overline{\varphi} : \kappa(\omega) \to \kappa(\nu)$ be the induced map. Then

$$\partial_\nu \circ \varphi_* = e \cdot \overline{\varphi}_* \circ \partial_\omega$$

R3b Let $\varphi : F \to E$ be finite and let ν be a valuation on F. For the extensions ω of ν to E, let $\varphi_\omega : \kappa(\nu) \to \kappa(\omega)$ be the induced maps. Then

$$\partial_\nu \circ \varphi^* = \sum_\omega \varphi_\omega^* \circ \partial_\omega.$$

R3c Let $\varphi : E \to F$ and let ν be a valuation on F which is trivial on E. Then $\partial_\nu \circ \varphi_* = 0$.

R3d Let $\varphi : E \to F$ and let ν be a valuation on F which is trivial on E. Let $\overline{\varphi} : E \to \kappa(\nu)$ be the induced map and let π be a prime of ν. Then $s_\nu^\pi \circ \varphi_* = \overline{\varphi}_*$.

R3e For a valuation ν on F, a ν-unit u and $\rho \in M(F)$ one has

$$\partial_\nu(\{u\} \cdot \rho) = -\{\overline{u}\} \cdot \partial_\nu(\rho).$$

The maps φ_* and φ^* are called the **restriction** and the **corestriction** homomorphisms, respectively. We use the notations $\varphi_* = r_{E/F}$ and $\varphi^* = c_{E/F}$ if there is no ambiguity. Note that they correspond to the restriction map (3.2) and the norm map (3.21), respectively.

Note that ∂_ν in the definition corresponds to the residue homomorphism (3.3). The map s_ν^π is known as the *specialization homomorphism*.

Corollary 3.28. *Choose $y = 1 \in \mathrm{K}_0^M(E)$ in R2c. Then for $\varphi : F \to E$, one has*

$$\varphi^* \circ \varphi_* = [E : F] \cdot \mathrm{id}\,.$$

*This follows from Corollary (*3.24*).*

Corollary 3.29. *By R1c, for finite totally inseparable $\varphi : F \to E$ one has*
$$\varphi_* \circ \varphi^* = \deg \varphi \cdot \mathrm{id}\,.$$

Corollary 3.30. *By R3e, for a valuation ν on F, $x \in \mathrm{K}_n^M(F)$, $\rho \in M(F)$ and a prime π of ν, one has*
$$\begin{aligned}\partial_\nu(x \cdot \rho) &= \partial_\nu(x) \cdot s_\nu^\pi(\rho) + (-1)^n s_\nu^\pi(x) \cdot \partial_\nu(\rho) + \{-1\} \cdot \partial_\nu(x) \cdot \partial_\nu(\rho),\\ s_\nu^\pi(x \cdot \rho) &= s_\nu^\pi(x) \cdot s_\nu^\pi(\rho).\end{aligned}$$
If π' is another prime and u is the ν-unit with $\pi' = \pi u$, then
$$s_\nu^{\pi'}(x) = s_\nu^\pi(x) - \{\overline{u}\} \cdot \partial_\nu(x).$$

One can transform the cycle premodules over X to another one over a different k-scheme X'. Explicitly, let $M : \mathfrak{F}_X \to \mathsf{Ab}$ be a cycle premodule over X and $f : X' \to X$ be a morphism of schemes. Then $f^*M : \mathfrak{F}_{X'} \to \mathfrak{F}_X \to \mathsf{Ab}$ is a cycle premodule over X', where $\mathfrak{F}_{X'}$ is obtained by taking the precomposition of objects and morphisms.

Definition 3.31. A **pairing** $M \times M' \to M''$ of cycle premodules over X is given by bilinear maps for each $F \in \mathfrak{F}_X$:
$$M(F) \times M'(F) \to M''(F),$$
where $(\rho, \mu) \in M(F) \times M'(F)$ is sent to $\rho \cdot \mu$, which respect the gradings and have the properties P1-P3:

P1 For $x \in \mathrm{K}_*^M(F)$, $\rho \in M(F)$, $\mu \in M'(F)$ one has
P1a $(x \cdot \rho) \cdot \mu = x \cdot (\rho \cdot \mu)$,
P1b $(\rho \cdot x) \cdot \mu = \rho \cdot (x \cdot \mu)$.
P2 For $\varphi : F \to E$, $\alpha \in M(F)$, $\beta \in M(E)$, $\rho \in M'(F)$, and $\mu \in M'(E)$, one has
P2a $\varphi_*(\alpha \cdot \rho) = \varphi_*(\alpha) \cdot \varphi_*(\rho)$,
P2b if φ is furthermore finite, then $\varphi^*(\varphi_*(\alpha) \cdot \mu) = \alpha \cdot \varphi^*(\mu)$,
P2c if φ is furthermore finite, then $\varphi^*(\beta \cdot \varphi_*(\rho)) = \varphi^*(\beta) \cdot \rho$.
P3 For a valuation ν on F, $\eta \in M_n(F)$, $\rho \in M'(F)$, and a prime π of ν, one has
$$\partial_\nu(\eta \cdot \rho) = \partial_\nu(\eta) \cdot s_\nu^\pi(\rho) + (-1)^n s_\nu^\pi(\eta) \cdot \partial_\nu(\rho) + \{-1\} \cdot \partial_\nu(\eta) \cdot \partial_\nu(\rho).$$

A ring structure on a cycle premodule M is a pairing $M \times M \to M$ which induces on each $M(F)$ an associative and anti-commutative ring structure.

Definition 3.32. A **homomorphism** $\omega : M \to M'$ **of cycle premodules over** X of even (resp. odd) type is given, for each $F \in \mathfrak{F}_{\mathfrak{X}}$, by homomorphisms
$$\omega_F : M(F) \to M'(F)$$
which are even (resp. odd) and which satisfy (with the signs corresponding to even (resp. odd) type):

(a) $\varphi_* \circ \omega_F = \omega_E \circ \varphi_*$,
(b) $\varphi^* \circ \omega_E = \omega_F \circ \varphi^*$,
(c) $\{a\} \cdot \omega_F(\rho) = \pm\omega_F(\{a\} \cdot \rho)$,
(d) $\partial_\nu \circ \omega_F = \pm\omega_{\kappa(\nu)} \circ \partial_\nu$.

Here $E, F \in \mathfrak{F}_X$, $\varphi : F \to E$, $\rho \in M(F)$, ν is a valuation on F and $\kappa(\nu)$ is its corresponding residue field of valuation ring, $a \in X$.

Example 3.33. If $a \in X$ is a unit, then it induces a homomorphism $\{a\} : M \to M$ by $\{a\}_F(\rho) = \{a_F\} \cdot \rho$ where $a_F \in F^*$ is the restriction of a.

We will end this section by introducing one of the main theorem of the paper. The proof will be delayed to §4.3.

Theorem 3.34. *Milnor K-theory, together with the residue homomorphisms (3.3), restrictions (3.2), norm maps (3.21), is a $\mathbb{Z}$-graded cycle premodule over any field k. Moreover, its ring structure (3.1) is compatible with the pairing (Definition 3.31).*

3.3. **Rost cycle modules.** Write M to be a cycle premodule over some k-scheme X. We denote $M(x) \coloneqq M(\kappa(x))$ for $x \in X$. If X is normal, then for $x \in X$ of codimension 1, $\mathcal{O}_{X,x}$ is a DVR (see [Har13, II. Proposition 6.3A]). Thus, we have a map

$$\partial_x \coloneqq \partial_\nu : M(\xi) \to M(x) \tag{3.6}$$

given by D4 of Definition 3.27, where ξ is a generic point of X. Denote $X^{(1)}$ by the set of all points of codimension 1 in X. Let $x, y \in X$, we define the map

$$\partial^x_y : M(x) \to M(y) \tag{3.7}$$

by

$$\partial^x_y = \begin{cases} 0 & , y \notin Z^{(1)}, \\ \sum_{z|y} c_{\kappa(z)/\kappa(y)} \circ \partial_z & , \text{else} \end{cases}$$

where $Z = \overline{\{x\}}$, $\widetilde{Z} \to Z$ is the normalization. z runs through the finitely many points of $\widetilde{Z}$ lying over y.

Definition 3.35. M is a **cycle module** over X if it satisfies the following conditions:

- FD **Finite support of divisors**: Let X' be a normal scheme over X and $\rho \in M(\xi)$, where ξ is a generic point of X'. Then $\partial_x(\rho) = 0$ for all but finitely many $x \in (X')^{(1)}$.
- C **Closedness**: Let X' be a variety over X such that $X' = \operatorname{Spec} R$ for some local ring R with Krull dimension 2. Then

$$0 = \sum_{x \in X^{(1)}} \partial^x_{x_0} \circ \partial^\xi_x : M(\xi) \to M(x_0),$$

  where ξ is a generic point in X' and x_0 is the only closed point of X'.

If X' in the definition is actually a normal variety satisfying the condition (FD), then we define

$$d = (\partial^\xi_x)_{x \in X^{(1)}} : M(\xi) \to \coprod_{x \in X^{(1)}} M(x). \tag{3.8}$$

For the rest of the section, for every field F over X, we write $\mathbb{A}^1_F = \operatorname{Spec} F[t]$ to be the corresponding affine line with local field $F(t)$.

Corollary 3.36. *Let M be a cycle module over X, then the following holds:*

- *FDL* ***Finite support of divisors on the line****: Let $\rho \in M(F(t))$, then $\partial_\nu(\rho) = 0$ for all but finitely many valuations ν of $F(t)$ over the field $F \in \mathfrak{F}_X$.*
- *WR* ***Weak reciprocity****: Let ∂_∞ be the residue map for the valuation of $F(t)/F$ at the infinity. Then $\partial_\infty(\ker d) = 0$, where d is defined in (3.8) with $X = \operatorname{Spec} F[t]$.*

Proof. Let $X' = \operatorname{Spec} F[t]$ in the condition (FD) of Definition 3.35, then (FDL) follows immediately. Set $\mathbb{A}^2_F = \operatorname{Spec} F[u,w]$. To prove the condition (WR), for any $\rho \in \ker d$, we set $\eta = \{w\} \cdot \varphi_*(\rho)$ for $\varphi : F(t) \to F(u,w)$, $\varphi(t) = w/v$ defined in Definition 3.27. By R2 (R2a-R2c) and R3 (R3a-R3e) in Definition 3.27, one obtains for any $y, z \in (\mathbb{A}^2_{\langle u,w\rangle})^{(1)}$ and $x \in (\mathbb{A}^2_{\langle u,w\rangle})^{(1)} \setminus \{y,z\}$,

$$\begin{aligned}
\partial_x(\eta) &= 0,\\
\partial_y(\eta) &= -\{w\} \cdot r_{\kappa(y)/F}(\partial_\infty(\rho)),\\
\partial_z(\eta) &= \partial_z(\{w\} \cdot \varphi_*(\rho)) = \partial_z(\{v\} \cdot \varphi_*(\rho) + \varphi_*(\{t\} \cdot \rho))\\
&= -\{v\} \cdot r_{\kappa(z)/F}(\partial_0(\rho)) + r_{\kappa(z)/F} \circ \partial_0(\{t\} \cdot \rho)
\end{aligned}$$

where each boundary operator on the left hand sides is defined in (3.6). and r is the restriction homomorphism (3.2). By condition (C) and $\partial_0(\rho) = 0$, we obtain

$$0 = \sum_{x \in (\mathbb{A}^2_{\langle u,w\rangle})^{(1)}} \partial_0^x \circ \partial_x(\eta) = \partial_0^y \circ \partial_y(\eta) = -\partial_\infty(\rho).$$

□

As one might imagine, the Milnor K-theory is an example of cycle module. We rephrase Theorem 3.34 in the following statement and defer the proof to §4.3 as well.

Theorem 3.37. *Milnor K-theory is a $\mathbb{Z}$-graded cycle module over any field k.*

The rest of the section devotes to the proof of the following proposition:

Proposition 3.38. *Let M be a cycle module over X. The following properties are satisfied:*

H ***$\mathbb{A}^1$-homotopy invariance****: The following sequence is exact:*

$$0 \to M(F) \xrightarrow{r_{F(t)/F}} M(F(t)) \xrightarrow{d} \coprod_{x \in \mathbb{A}^1_F} M(x) \to 0.$$

Here the map d is defined in (3.8).

RC ***Reciprocity for curves****: Let Y be a proper scheme of dimension 1 over F, then for any generic point $\xi \in Y$, the sequence*

$$M(\xi) \xrightarrow{d} \coprod_{x \in X} M(x) \xrightarrow{c} M(F)$$

satisfies $c \circ d = 0$, where $c = \sum c_{\kappa(x)/F}$.

Proof. (a) For the property (**H**):

We need to show $d \circ r_{F(t)/F} = 0$, and $\ker d = \operatorname{Im} r_{F(t)/F}$. The former follows from R3c in Definition 3.27. It suffices to show the exactness, or only $\ker d \subset \operatorname{Im} r_{F(t)/F}$.

Let $E = F(t)$, and $i, \varphi : E \to E(u)$ be the homomorphisms over F with $i(t) = t$, $\varphi(t) = t+u$. For any $\rho \in \ker d$, put

$$\eta = \{u\} \cdot (\rho(t+u) - \rho(u)) = \{u\} \cdot (\varphi_*(\rho) - i_*(\rho)) \in M(F(t)(u)).$$

For each valuation ν of $E(u)$ over E, by R3c and R3e,

$$\begin{aligned}\partial_\nu(\eta) &= -\{\overline{u}\} \cdot \partial_\nu(\varphi_*(\rho) - i_*(\rho)) \\ &= 0\end{aligned}$$

For all $\nu \neq 0, \infty$. The case of 0 is also ruled out by R3d since the valuation at $u = 0$ restricts trivially under i and φ. On the other hand, by (WR) in Corollary 3.36, $\partial_\infty(\eta) = 0$. By R3d and R3e, one also has

$$\begin{aligned}&\partial_\infty(\{u\} \cdot i_*(\rho)) = -\rho, \\ &\partial_\infty(\{u\} \cdot \varphi_*(\rho)) = \partial_\infty(\{u/(t+u)\} + \{t+u\} \cdot \varphi_*(\rho)) \\ &\quad = -\{\overline{u/(t+u)}\} \cdot \partial_\infty(\varphi_*(\rho)) + \partial_\infty(\varphi_*(\{t\} \cdot \rho)) \\ &\quad = 0 + r_{E/F}(\partial_\infty(\{t\} \cdot \rho)).\end{aligned}$$

It follows that

$$\rho = r_{E/F}(\partial_\infty(\{t\} \cdot \rho)) \in r_{E/F}(M(F)) = \operatorname{Im} r_{F(t)/F}.$$

(b) For the property (**RC**):

By assumption, there is a finite morphism $Y \to \mathbb{P}^1$ over F. By R3b, we can reduce to the case $Y = \mathbb{P}^1$. It suffices to check

$$\sum_{y \in \mathbb{P}^1} c_{\kappa(y)/F} \circ \partial_y \circ \Phi^y = 0, \tag{3.9}$$

where $\Phi^y : M(y) \to M(F(t))$ with

$$\Phi^y(\rho) = c_{\kappa(y)(t)/F(t)}(\{t - t(y)\} \cdot r_{\kappa(y)(t)/\kappa(y)}(\rho)).$$

By R3b-R3e, $d \circ \sum_y \Phi^y = 0$. On the other hand, one has the following by R3b and R3d:

$$\partial_\infty \circ \Phi^y = -c_{\kappa(y)/F}. \tag{3.10}$$

The equality (3.9) now follows directly from $d \circ \sum_y \Phi^y = 0$ and (3.10).

□

4. Cycle complexes

In this chapter, we put together the knowledge of Chow groups, Milnor K-theory, Rost cycle modules into a certain chain complex, known as the Rost *cycle complex*. This will the show the deep connection between these three important concepts, bridging the realm of algebraic geometry and number theory, as well as homological algebra.

4.1. **Cycle complexes and properties.** Let M be a cycle module over some nice k-scheme X, and N be a cycle module over some equally nice k-scheme Y. Set $U \subset X$ and $V \subset Y$. Write $\alpha^x_y : M(x) \to N(y)$ for the component of

$$\alpha : \coprod_{x \in U} M(x) \to \coprod_{y \in V} N(y).$$

If $\omega : M \to N$ is a homomorphism of cycle modules over X, we write

$$\omega_\# : \coprod_{x \in U} M(x) \to \coprod_{x \in U} N(x), \tag{4.1}$$

and observe that

$$(\omega_\#)^x_y = \begin{cases} \omega_{\kappa(x)}, & x = y, \\ 0, & x \neq y. \end{cases}$$

Definition 4.1. Let M be a cycle module over X as above. For each integer n, we write

$$C_n(X; M) \coloneqq \coprod_{x \in X_{(n)}} M(x),$$

where $X_{(n)}$ is the set of points of X of dimension n. Define

$$d = d_X : C_n(X; M) \to C_{n-1}(X; M)$$

by $d^x_y = \partial^x_y$ as in (3.7). The tuple $(C_*(X; M), d)$ is a chain complex, called (Rost) **cycle complex on X with coefficient in M**.

Proposition 4.2. *This definition is valid, i.e. $d^2 = 0$.*

Proof. This is immediately from (C) in Definition 3.35. □

In the homological algebra, we usually care about the maps between different complexes induced by the map of schemes. Once such maps are established, we want to study the long exact sequence induced by maps of complexes. These are the goals for the rest of section. We fix a cycle module M for now.

4.1.1. *Pushforward.*

Definition 4.3. Let $f : X \to Y$ be a proper morphism of schemes. The **pushforward** of cycle complexes

$$f_* : C_n(X; M) \to C_n(Y; M)$$

is defined by

$$(f_*)^x_y = \begin{cases} c_{\kappa(x)/\kappa(y)}, & y = f(x) \text{ and } [\kappa(x) : \kappa(y)] < \infty, \\ 0, & \text{else.} \end{cases}$$

4.1.2. *Pullback.* The construction of pullback requires more efforts. Let $g : Y \to X$ be a (flat) morphism of schemes. We denote

$$s(g) = \max\{\dim(y) - \dim(g(y)) : y \in Y\}. \tag{4.2}$$

For $x \in X$, we write $Y_x = Y \times_X \operatorname{Spec} \kappa(x)$. So if $x \in X_{(p)}$, $y \in Y_{(q)}$, $g(y) = x$ and $s(g) \leq q - p$, then $y \in Y^{(0)}_x$.

Now let $\mathcal{F}$ be a coherent sheaf over Y, $x \in X$ and $y \in Y^{(0)}_x$. We define the following integer

$$[\mathcal{F}, g]^x_y = \ell_R(\widetilde{\mathcal{F}}), \tag{4.3}$$

where $\widetilde{\mathcal{F}}$ is the pullback of $\mathcal{F}$ along $Y_{x,(y)} \to Y_x \to Y$ for $Y_{x,(y)}$ the localization of Y_x at y, and ℓ_R is the length of $\widetilde{\mathcal{F}}$ considered as R-module for some R with $\operatorname{Spec} R = Y_{x,(y)}$. Fix $s \in \mathbb{Z}$. We impose another requirement for $g : Y \to X$: $s(g) \leq s$.

Definition 4.4. Let $g : Y \to X$ be defined as above with $s(g) \le s$, and $\mathcal{F}$ be a coherent sheaf over Y. The **pullback** of cycle complexes

$$[\mathcal{F}, g, s] : C_n(X; M) \to C_{n+s}(Y; M)$$

is defined by

$$[\mathcal{F}, g, s]^x_y = \begin{cases} [\mathcal{F}, g]^x_y \cdot r_{\kappa(y)/\kappa(x)}, & g(y) = x, \\ 0, & \text{else.} \end{cases}$$

Here the notion $[\mathcal{F}, g]$ is defined in (4.3).

Proposition 4.5. *Let F be a field and $g : Y \to \operatorname{Spec} F$ be a morphism with $s(g) \le s$. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be a short exact sequence of coherent sheaves over Y. Then*

$$[\mathcal{F}', g, s] - [\mathcal{F}, g, s] + [\mathcal{F}'', g, s] = 0.$$

Proof. Observe that, as R-module for some R with $\operatorname{Spec} R = Y_{x,(y)}$ and $y \in Y^{(0)}_x$, the length of $\mathcal{F}$ is sum of ones of $\mathcal{F}'$ and $\mathcal{F}''$. The proposition follows. $\square$

If $g : Y \to X$ in Definition 4.4 is said to be of **constant relative dimension** s, if all fibers are equidimensional of s or empty. In this case, we write (for $\dim g = s$)

$$g^* := [\mathcal{O}_Y, g, \dim(g)]. \tag{4.4}$$

4.1.3. *Long exact sequence.* Let X be a nice k-scheme, $i : Y \to X$ be a closed immersion, and $j : X \to Z = X \backslash Y$ be the projection of the open complement.

Definition 4.6. For the tuple (Y, X, Z), we have a long exact sequence of cycle complexes

$$\cdots \to C_n(Y; M) \xrightarrow{i_*} C_n(X; M) \xrightarrow{j_*} C_n(Z; M) \xrightarrow{\partial} C_{n-1}(Y; M) \to \cdots,$$

where the all arrows except for ∂ are pushforwards in Definition 4.3, and ∂ is the **boundary map** defined via (3.7)

$$\partial := \partial^x_y$$

for $x, y \in Z \subset X$.

4.1.4. *Grading.* Recall that a cycle module M admits a $\mathbb{Z}/2$-grading or a $\mathbb{Z}$-grading (Definition 3.27). The interaction of such gradings and the grading in Definition 4.1 needs further clarification. Before we discuss this issue, it will be appropriate the bring up the following fact:

Proposition 4.7. *The cycle complex $C_n(X; M)$ is a $\mathrm{K}^M_*(F)$-module for some field $F \subset \mathcal{O}^*_X = \Gamma(X, \mathcal{O}_X)$ where X is over.*

Proof. Let $a_1, \cdots, a_n$ be the global sections for the structure sheaf $\mathcal{O}_X$. We define a homomorphism

$$\{a_1, \cdots, a_n\} : C_n(X; M) \to C_n(X; M)$$

by

$$\{a_1, \cdots, a_n\}^x_y(\rho) = \begin{cases} \{a_1(x), \cdots, a_n(x)\} \cdot \rho, & y = x, \\ 0, & \text{else.} \end{cases}$$

If X is defined over F, then $\{a_1(x), \cdots, a_n(x)\} \in \mathrm{K}^M_*(F)$, the result follows. $\square$

For the case of $\mathbb{Z}/2$-grading $M = M_0 \oplus M_1$, the $\mathbb{Z}/2$-grading of M induces a $\mathbb{Z}/2$-grading on $C_*(X; M)$ by

$$C_n(X; M, i) = \coprod_{x \in X_{(n)}} M_{n+i \mod 2}(x),$$

where $i = 0, 1$. Suppose $\alpha : X \to Y$ be a map of schemes that respects the grading, i.e.

$$\alpha(C_*(X; M, i)) \subset C_*(Y; M, i + j),$$

where $j = 0, 1$. Write $\mathrm{sgn}(\alpha) = (-1)^j$. It follows that

- $\mathrm{sgn}(f_*) = \mathrm{sgn}(g^*) = 1$, for f, g in Definition 4.3 and (4.4), respectively.
- $\mathrm{sgn}(\{a_1, \cdots, a_n\}) = (-1)^n$, for $a_1, \cdots, a_n$ the global sections for the structure sheaf $\mathcal{O}_X$.
- $\mathrm{sgn}(\partial) = -1$, for ∂ in Definition 4.6.

Moreover, if we write $\delta(\alpha) = d \circ \alpha - \mathrm{sgn}(\alpha) \cdot \alpha \circ d$, then $\mathrm{sgn}(\delta(\alpha)) = -\mathrm{sgn}(\alpha)$, $\delta^2(\alpha) = 0$, and $\delta(\alpha \circ \beta) = \delta(\alpha) \circ \beta + \mathrm{sgn}(\alpha) \cdot \alpha \circ \delta(\beta)$. So all the maps of concern will respect the $\mathbb{Z}/2$-grading.

For the case of $\mathbb{Z}$-grading $M = \coprod M_i$, we can similarly obtain an induced $\mathbb{Z}$-grading on the complex by

$$C_n(X; M, i) = \coprod_{x \in X_{(n)}} M_{n+i}(x).$$

Then there is decomposition of complexes

$$C_n(X; M) = \coprod_{i \in \mathbb{Z}} C_n(X; M, i).$$

This reflects a natural $\mathbb{Z} \times \mathbb{Z}$-grading of the cycle complex. In practice, this barely change anything. So, we can only focus on the $\mathbb{Z}/2$-grading.

4.2. **Compatibility.** This section is designed to study the interactions between the operations defined in the last section. In particular, we are interested in the relations between pushforwards and pullbacks. We will state without proof about the relations between pushforwards/pullbacks with differentials/boundaries. At the end of this section, we will try establish the bridge between cycle complexes and Chow groups that studied in §2.3. The eager reader is encouraged to jump straight to the conclusion, with little loss in skipping the proof.

Proposition 4.8. *Let X, Y, Z be nice schemes. Fix a cycle module M whenever it is needed.*

(a) Let $f : X \to Y$, $f' : Y \to Z$ be proper morphisms. Then $(f' \circ f)_* = f'_* \circ f_*$.

(b) Let $g : Y \to X$ and $g' : Z \to Y$ be morphisms. Suppose $s \geq s(g)$ and $s' \geq s(g')$. Let $\mathcal{F}, \mathcal{F}'$ be coherent sheaves on Y, Z, respectively, with $\mathcal{F}'$ flat over Y. Then $s + s' \geq s(g \circ g')$ and

$$[(g')^*\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{F}', g \circ g', s + s'] = [\mathcal{F}', g', s'] \circ [\mathcal{F}, g, s].$$

Moreover, $(g \circ g')^* = (g')^* \circ g^*$ for g', g satisfy the conditions in (4.4) with g' flat.

(c) Consider the pullback diagram

$$\begin{array}{ccc} U & \xrightarrow{g'} & Z \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y & \xrightarrow{g} & X \end{array}$$

with f, f' proper. Suppose $s \geq s(g)$ and $s \geq s(g')$. Let $\mathcal{F}$ be a coherent sheaf over Y. Then

$$[\mathcal{F}, g, s] \circ f_* = (f')_* \circ [(f')^*\mathcal{F}, g', s].$$

Moreover, if g satisfies the conditions in (4.4), then

$$g^* \circ f_* = (f')_* \circ (g')^*.$$

Proof. (a) This is immediate from R1a in Definition 3.27.

(b) $s + s' \geq s(g \circ g')$ is direct from the definition (4.2). Let $x \in X$, $y \in Y_x$, $z \in Z_y$ with $\dim(y) = \dim(x) + s$, and $\dim(z) = \dim(y) + s'$. It suffices to check

$$[(g')^*\mathcal{F} \otimes_{\mathcal{O}_Z} \mathcal{F}', g \circ g']^x_z = [\mathcal{F}', g']^y_z \circ [\mathcal{F}, g]^x_y.$$

From (4.3), we can assume $X = \operatorname{Spec} \kappa(x)$ and $Y = \operatorname{Spec} R$ for some R. By dévissage from the flatness of $\mathcal{F}'$ over R and Proposition 4.5, we can pass to the case $\mathcal{F} = \kappa(y)$. The result then follows.

(c) Let n be any non-negative integer. Define $\delta = [\mathcal{F}, g, s] \circ f_* - (f')_* \circ [(f')^*\mathcal{F}, g', s]$. It suffices to show $\delta^z_y = 0$ for each $z \in Z_{(n)}$ and $y \in Y_{n+s}$. If $g(y) \neq f(z)$, then $\delta^z_y = 0$ is clearly zero. Suppose $g(y) = f(z) = x$. By the same reason as in the proof of (b), $\dim(z) \geq \dim(z)$ and $\dim(x) \geq \dim(y) - s(g) \geq n$. It follows that $\dim(x) = \dim(z) = n$, and so $\kappa(z)$ is finite over $\kappa(x)$.

Let $u \in U_z$ be a maximal point of the fiber over z. We observe that $\dim(u) \geq \dim(y) = n+s$ and $\dim(u) \leq \dim(z) + s(g') \leq n + s$. Thus, $u \in U_{(n+s)}$, implying that δ^z_y will not change if we replace X by $\operatorname{Spec} \kappa(x)$, Z by $\operatorname{Spec} \kappa(z)$, and Y by $Y_{x,(y)} = \operatorname{Spec} R$ for some R, as well as n, s by 0. So f is finite and flat. By dévissage from the flatness of $\mathcal{F}$ over R and Proposition 4.5, we may reduce to the case $\mathcal{F} = \kappa(y)$ as R-module. The question then turns into the proof for the case $Y = \operatorname{Spec} \kappa(y)$, which is clear by R1c in Definition 3.27.

□

Proposition 4.9. *Let $f : Y \to X$ be proper.*

(a) If a is a unit on X, then

$$f_* \circ \{f^*(a)\} = \{a\} \circ f_*.$$

(b) If f is finite and flat, and a is a unit on Y, then

$$f_* \circ \{a\} \circ f^* = \{\widetilde{f}_*(a)\},$$

where $\widetilde{f}_* : \mathcal{O}^*_Y \to \mathcal{O}^*_X$ is the induced morphism on global sections of the structure sheaves.

Proof. (a) This is immediate from R1b in Definition 3.27.

(b) Reduce to the case $X = \operatorname{Spec} F$ for F a field. By R1c in Definition 3.27, we obtain

$$\begin{aligned} f_* \circ \{a\} \circ f^* &= \sum_{y \in Y} \ell(\mathcal{O}_{Y,y}) \cdot c_{\kappa(y)/F}(\{a(y)\}) = \sum_{y \in Y} c_{\kappa(y)/F}(\{a(y)\}) \\ &= \sum_{y \in Y} -\partial_{y,\nu_\infty}(\{a(y)\}) = \sum_{y \in Y} \deg(\widetilde{f}_*) \cdot \{\widetilde{f}_*(a)(y)\} \qquad \text{(by Definition 3.21)} \\ &= \{\widetilde{f}_*(a)\} \end{aligned}$$

as desired.

□

For completeness, we now state some propositions without proving them. The detailed treatment can be found in [Ros96, §4].

Proposition 4.10 (Rost, [Ros96, Lemma 4.3])**.** *Let a be a unit on X.*

(a) For $g : Y \to X$ satisfies the conditions in (4.4), one has

$$g^* \circ \{a\} = \{g^*(a)\} \circ g^*.$$

(b) For the tuple (Y, X, Z) as in Definition 4.6, one has

$$\partial \circ \{j^*(\alpha)\} = -\{i^*(a)\} \circ \partial$$

Proposition 4.11 (Rost, [Ros96, Proposition 4.6])**.** *The following statements hold:*

(a) For $f : X \to Y$ a proper map, one has

$$d_Y \circ f_* = f_* \circ d_X.$$

d_X and d_Y are the same as ones in Definition 4.1.

(b) Let $g : Y \to X$ be a morphism satisfies the conditions in (4.4), and $\mathcal{F}$ be a coherent sheaf on Y flat over X. Then for $s \geq s(g)$,

$$d_Y \circ [\mathcal{F}, g, s] = [\mathcal{F}, g, s] \circ d_X.$$

Moreover, for g flat,

$$g^* \circ d_X = d_Y \circ g^*.$$

(c) For a unit a on X, one has

$$d_X \circ \{a\} = -\{a\} \circ d_X.$$

(d) For the tuple (Y, X, Z) as in Definition 4.6, one has

$$d_Y \circ \partial = -\partial \circ d_Z.$$

The final part of this section will refocus attention on the Chow groups. Fix a cycle module M. Before we discover the connections, it is convenient that we introduce the "dual" version of the cycle complexes. Namely, we write

$$C^n(X; M) \coloneqq \coprod_{x \in X^{(n)}} M(x), \tag{4.5}$$

and define

$$d = d^X : C^n(X; M) \to C^{n+1}(X; M)$$

by $d^x_y = \partial^x_y$ as in (3.7). It will be not hard to show $d^2 = 0$, similar to Proposition 4.2. Compared to Definition 4.1, the only difference is to choose the points of codimension n, instead of dimension n. So, if X is equidimensional of degree d, then $X^{(n)} = X_{(d-n)}$, and so $C^n(X; M) = C_{d-n}(X; M)$. The treatment of $\mathbb{Z}/2$-grading or $\mathbb{Z}$-grading of M is basically the same.

Definition 4.12 (Rost)**.** The **Chow group with coefficients in** M is defined as the homology group of the complex $C_*(X; M)$, denoted $CH_*(X; M)$. Similarly, we use the notion $CH^*(X; M)$ to mean the homology of $C^*(X; M)$.

The morphisms (pushforwards, pullbacks, units, and boundary maps) and their properties between Chow groups in coefficients M are already discussed in §4.1 and previous context in §4.2. To see how Chow groups in coefficients are related to the classical Chow groups, note that we can decompose

$$C_*(X;M) = \coprod_{n\in\mathbb{Z}} C_*(X;M,n),$$

and by choosing $M = \mathrm{K}_*^M$, we obtain

$$CH_n(X) = H_n(C_*(X;\mathrm{K}_*^M,-n)).$$

Similarly, we obtain

$$CH^n(X) = H_n(C^*(X;\mathrm{K}_*^M,n)).$$

These are from (3.5) at the beginning of §3.2.

4.3. **Milnor K-theory as cycle module.** This section is devoted to give a sketch proof of Theorem 3.37.

Theorem 4.13 (Theorem 3.37)**.** *Milnor K-theory is a $\mathbb{Z}$-graded cycle module over any field k.*

We have already defined residue homomorphisms (3.3), restrictions (3.2), norm maps (3.21). It suffices to check they satisfy the desired properties R1-R3 in Definition 3.27 and (FD), (C) in Definition 3.35. Let $\varphi : F \to E$ and $\psi : E \to L$ be finite field extensions. Here fields E, F, L are elements in $\mathfrak{F}_X$. The data D1-D4 in Definition 3.27 is easy to check:

- We ask $\varphi_* = r_{E/F}$, so $\varphi_* : \mathrm{K}_*^M(F) \to \mathrm{K}_*^M(E)$ sends $\{a_1, \cdots, a_n\}$ to $\{\varphi(a_1), \cdots, \varphi(a_n)\}$. This is clearly of degree 0.
- We ask $\varphi^* = c_{E/F}$, so $\varphi_* : \mathrm{K}_*^M(E) \to \mathrm{K}_*^M(F)$ is defined in Definition 3.21. This is also of degree 0.
- $\mathrm{K}_*^M(F)$ is clearly a $\mathrm{K}_*^M(F)$-module itself with the module structure given by the ordinary multiplication.
- We ask $\partial_\nu = \partial_{x,\nu}$ (omit one of the indices if applicable) as in (3.3). This is clearly of degree -1.

It is not hard to see R1a, R1b, R2a hold. R2b and R2c follows from the projection formula (Proposition 3.23).

Proposition 4.14. *The base change formula R1c holds.*

Proof. This follows from Kato's lemma. See [Wei13, III. Proposition 7.6.4]. □

Let ν be a discrete valuation. For a prime π of ν on F, we define

$$s_\nu^\pi : \mathrm{K}_*^M(F) \to \mathrm{K}_*^M(\kappa(\nu)) \tag{4.6}$$

by

$$s_\nu^\pi(x) = \partial_\nu(\{-\pi\} \cdot x).$$

This is clearly a ring homomorphism because ∂_ν is. Actually, this is desired specialization homomorphism for the Milnor K-theory that is used in R3.

Proposition 4.15. *R3a holds.*

Proof. Let $\varphi : E \to F$ and ν be a discrete valuation on F which restricts to a nontrivial valuation ω on E with ramification index e. It follows that $\mathfrak{m}_\omega \mathcal{O}_\nu = \mathfrak{m}_\nu^e$. Consider the following diagram

$$\begin{tikzcd} \mathrm{K}_n^M(F) \arrow[r, "\partial_\nu"] & \mathrm{K}_n^M(\kappa(\nu)) \\ \mathrm{K}_n^M(E) \arrow[r, "\partial_\omega"] \arrow[u, "r_{F/E}"] & \mathrm{K}_n^M(\kappa(\omega)) \arrow[u, "e\cdot r_{\kappa(\nu)/\kappa(\omega)}"'] \end{tikzcd}$$

The diagram commutes because for $a_1, \cdots, a_{n-1} \in \mathcal{O}_\omega^*$ and $x \in F^*$,

$$\partial_\nu(\{a_1, \cdots, a_{n-1}, x\}) = \nu(x) \cdot \{\overline{a_1}, \cdots, \overline{a_{n-1}}\} = e \cdot \omega(x) \cdot \{\overline{a_1}, \cdots, \overline{a_{n-1}}\}. \tag{4.7}$$

Writing $\overline{\varphi} = r_{\kappa(\nu)/\kappa(\omega)}$, we obtain our desired result. $\square$

Corollary 4.16. *R3c holds.*

Proof. In (4.7), taking $\omega(x) = 0$, the result follows. $\square$

Proposition 4.17. *R3b holds.*

Proof. This follows from [Wei13, III. Exercise 7.9]. For a detailed deduction, see the proof of [Hes05, Proposition 15]. $\square$

Proposition 4.18. *R3d and R3e hold.*

Proof. Let $\varphi : E \to F$ be a finite field extension. One sees for $\{a_1, \cdots, a_n\} \in \mathrm{K}_n^M(E)$,

$$\begin{aligned} s_\nu^\pi(r_{F/E}(\{a_1, \cdots, a_n\})) &= s_\nu^\pi(\{\varphi(a_1), \cdots, \varphi(a_n)\}) \\ &= \partial_\nu(\{-\pi\} \cdot \{\varphi(a_1), \cdots, \varphi(a_n)\}) \\ &= \nu(-\pi) \cdot \{\overline{\varphi(a_1)}, \cdots, \overline{\varphi(a_n)}\} \\ &= \{\overline{\varphi(a_1)}, \cdots, \overline{\varphi(a_n)}\} = \overline{\varphi}_*(\{a_1, \cdots, a_n\}). \end{aligned}$$

R3e follows directly by a similar computation applying to $\{a_1, \cdots, a_n\} \in \mathrm{K}_n^M(E)$. We encourage the serious readers to verify it for yourselves. $\square$

There is an obvious $\mathbb{Z}$-grading on the Milnor K-theory from Definition 3.1. At this stage, we've shown that the Milnor K-theory is a cycle premodule. To prove this is a cycle module, we still need to prove the conditions (FD) and (C) in Definition 3.35.

Proposition 4.19. *Conditions (FD) and (C) hold.*

Proof. (FD) follows from as for classical divisors. (C) is proved in [Kat96, §3, Theorem 1]. $\square$

Combining all propositions we proved Theorem 3.37 (and in particular Theorem 3.34). Hence, the following complex is well-defined:

$$\cdots \to \coprod_{y \in X_{(i+1)}} \mathrm{K}_*^M(\kappa(y)) \xrightarrow{d} \coprod_{y \in X_{(i)}} \mathrm{K}_*^M(\kappa(y)) \xrightarrow{d} \coprod_{y \in X_{(i-1)}} \mathrm{K}_*^M(\kappa(y)) \to \cdots$$

which is in line with the goal of our paper. We have now achieved the result we hoped for in §1.

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